

# Reflections Around the Ramanujan Centenary

## *Atle Selberg*



The Norwegian Mathematician Atle Selberg is considered to be one of the great number theorists of this century. He was awarded the Fields Medal in 1950 and is currently at the Institute for Advanced Study, Princeton, USA.

**This article is based on the tape of an extemporaneous talk to a general audience at the Tata Institute of Fundamental Research, Bombay, India, given after the conclusion of the Centenary Conference held there (January 1988).<sup>1</sup>**

Srinivasa Ramanujan's work played a very important role in my own development as a mathematician. I first saw his name in 1934, when I came across an article in the periodical of the Norwegian Mathematical Society. It was a periodical that my father subscribed to and the title of the article was, if I translate it from Norwegian, "The Indian Srinivasa Ramanujan, a remarkable mathematical genius". But I should add that, since there is no one-to-one correspondence between languages, for the word I have translated here as "remarkable", the Norwegian word carries also a connotation meaning "unusual and somewhat strange". It was written by a professor of mathematics at the University of Oslo by the name of Carl Störmer who had actually started out by being interested in Number Theory in his youth, later turned to the mathematical theory of the aurora borealis (northern lights) and made quite a name for himself in connection with that. But he retained his interest in mathematics (Pure mathematics).

The material for his article was mainly taken from the biographical articles in the *Collected Papers* of Ramanujan which had been published in 1927 by the Cambridge University Press. It gave a sketch of the history of Ramanujan's life and he quoted quite a number of his results and samples of what I thought were extremely remarkable, strange, and beautiful formulas. It was, if I remember, probably about 15 pages long and not more than 20

<sup>1</sup> Reproduced from *Atle Selberg, Collected Papers. Vol I*, pages 695-701, Springer Verlag, 1989.



pages. But it made a very deep and lasting impression on me and it fascinated me very much.

At that time, I was still a school boy and, for some years, I had been reading mathematics on my own in a rather unsystematic and haphazard way, in my father's mathematical library which was quite large for a private individual, I may say.

I may make a digression on that, because it shows how chance plays a role in one's life - a very great role often. I had started reading mathematics when I was about thirteen or so. I had accidentally opened a book and come across Leibnitz's series for  $\pi/4$ :  $1 - 1/3 + 1/5 - 1/7$  and so forth involving reciprocals of the odd integers with alternating signs. Till then, school mathematics had always bored me but this seemed such a very strange and beautiful relationship that I determined I would read that book in order to find out how this formula came about.

It is a wonder that I did not stop because the book actually started with a long chapter on the concept of the real number and with the Dedekind cut which was not the most inspiring beginning.

At any rate, when I saw the article about Ramanujan, I had already made up my mind to go into mathematics but I had no more than vague notions about what kind of mathematics. I think I was at that time mostly thinking of going into the general theory of analytic functions; more specifically, something like the Nevanlinna theory which was then very much in the foreground and which actually my oldest brother had started doing. He was then a research fellow at the university. Another brother who was already a student of mathematics for a few years was grappling with the Theory of Numbers. He had also read this article on Ramanujan, and so he borrowed from the university library Ramanujan's *Collected Papers* and later he brought the book home during a vacation.

So I got a chance to browse through it for several weeks. It



seemed quite like a revelation - a completely new world to me, quite different from any mathematics book I had ever seen - with much more appeal to the imagination, I must say. And frankly, it still seems very exciting to me and also retains that air of mystery which I felt at the time. It was really what gave the impetus which started my own mathematical work. I began on my own, experimenting with what is often referred to as q-series and identities and playing around with them.

In the summer of 1935, as I finished the gymnasium, I wrote a manuscript - I will give the title in English, *On some arithmetical identities*. German was my best foreign language at that time and I gave this manuscript to Professor Störmer in Oslo when I started my university studies at the beginning of the fall term in 1935. He sent it to Professor G N Watson in Birmingham, England, in order that it could be refereed. He had some connections with Watson before. And Watson, after having kept it for quite some time (much too long I thought at that time but now that I myself have been a referee, I have somewhat more understanding for Watson's delay!), finally sent it back with the recommendation that it be published, which it duly was in 1936.

Watson also sent me a lot of his reprints, and particularly, reprints dealing with things from Ramanujan's *Notebooks* and *Nachlass* or posthumous papers. Among these were, for instance, reprints of Watson's papers on the mock-theta functions of orders 3 and 5 where he proves the various identities or relations that Ramanujan had stated for these functions and also showed that they had the required very precise asymptotic behaviour when one approaches the roots of unity on the unit circle. And this motivated my second paper which dealt with mock-theta functions of order 7 and established their asymptotic behaviour.

In the meantime, I had acquired my own copy of Ramanujan's *Collected Papers* which was a present from my father that I carry with me now. I also thought of studying some of the other papers - first of all, the joint paper with Hardy on the partition function

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expressing the number of ways in which a number can be written as a sum of integers. The paper is entitled *Asymptotic formulae in combinatory analysis*. The other papers involved modular forms and their coefficients. But this was my first contact with the theory of modular forms and automorphic forms in general, as well as with discrete or -as we in general called it at that time - discontinuous groups and this is a theme that ever since has remained one of the main interests, perhaps the main interest in my mathematical work.

Before I go on to Ramanujan's mathematics, let me say something about mathematics or mathematicians in general, for the benefit of the non-mathematicians. Let me interject that I feel truly sorry for them. I think they are missing what I think is the most exciting and rewarding intellectual activity.

Mathematics is often compared to the arts, particularly to music, and it is true that in mathematics as in music talent may flower at an astonishingly early age, though it has to be admitted, earlier in music than in mathematics. In mathematics, aesthetic considerations, beauty, simplicity and elegance are very important as well as truth. If one looks at mathematics as a body of knowledge, I think it definitely can be characterized as a science, but if one looks at the way in which it grows and accumulates, the actual doing of mathematics seems much more to be an art. Mathematics is concerned exclusively with objects and structures which are creations of the mind, although they may be suggested by or patterned on things that are found in the so-called real world. And since it deals only with creations of the mind, it is cumulative in a way that the other sciences are not.

Mathematics does not shed the old substances as new is added, in the way the natural sciences, for instance, will do. The work of Euclid, Apollonius or Archimedes, to mention some Greek mathematicians from antiquity, is as valid today as when it was done more than two millennia ago. But while the content or substance remains, the form in which it is presented is ever



changing. What we may refer to as the landscape of mathematics may change profoundly from one generation to another, and even during shorter time spans, fundamental changes may occur.

If I consider my own time as an active mathematician, the change between how I thought in the thirties and now, of course, is tremendous. The papers that appear today could not have been understood by anyone - or at least most of them - at that time; they often deal with concepts that did not exist then. Mathematics grows in many ways by diversification and complexification or specialization, as one subject may branch out in many directions and form separate specialities. On the other hand, we also have convergence and synthesis, simplification and even unification, as different fields of mathematics that seem distant and without any connections, may develop several bridges and eventually become closely interconnected. While Indian mathematics seems to have been built on the concept of the number from quite an early stage, Greek mathematics was built on the concept of point, line and plane and the relations between these; only after the revival of mathematics in Italy, was Western mathematics primarily based on the concept of number and remained so for several hundred years. Today, we have a mathematics which is primarily concerned with structure and relationships between structures rather than just relationships between numbers. The first beginnings of this we find around 1800 and the first breakthrough in this direction was, of course, the introduction of the abstract group concept. Now it is almost all-pervasive in the field of mathematics.

Mathematical talent is also something that occurs in many varieties. Some mathematicians are theory-builders, some are problem-solvers and some originate problems - I will not say that they create problems! They originate them. Or, they may produce the first isolated examples of new mathematical objects or relationships which later give rise to comprehensive theories. None of these different abilities or talents should be ranked

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Ramanujan's particular talent will seem to be primarily of an algebraic and combinatorial nature. He developed it, for a long time in complete isolation really without any contact with other mathematicians. He had, on his own, acquired an extraordinary skill of manipulation of algorithms, series, continued fractions and so forth, which certainly is completely unequalled in modern times. He seems also to have often taken a very particular delight in the special rather than the general, and if one looks in his *Notebooks*, and also in his letters to Hardy, for instance, he will often state or choose to state a special case or special cases that seem particularly striking, where very clearly he has much more general results underlying them.

He might very well have become a theory-builder, if he had a different and more conventional start and training as a mathematician. Even then, in what has been left in his work, there seems quite clear evidence that he had developed, on his own, a theory of modular forms and equations, for instance, but the precise form of this theory has to be guessed from the isolated results he wrote down in the *Notebooks*. What we have from him are mostly his many and often mysterious results and assertions in the *Notebooks*, and we have his published papers.

It is interesting to see how his published work has been assessed over the years. In the early years, it was clearly the joint work with Hardy on the partition function that drew most of the attention. When his *Collected Papers* appeared, and J E Littlewood wrote a review, he singled this paper out. Most of his review is devoted to trying to analyse how this paper arose. No doubt it was an extremely important paper both in itself and for the powerful tool, the circle method, which was for the first time introduced in Analytic Number Theory. But when it comes to Littlewood's account, it seems to me that it has to be quite wrong on several points.<sup>2</sup>

<sup>2</sup> See the Appendix to this talk in *Atle Selberg, Collected Papers*, Vol I, pages 701-706 concerning this and the following comments on the paper on the partition function.



Littlewood, by the way, wrote his review before the final word had been said about the partition function by H Rademacher's paper in 1937, when he found the exact formula. The paper by Hardy and Ramanujan contained surely a result that was very remarkable in itself; since  $p(n)$  is an integer, it allows exact computation on it. But it was not an exact formula. It was a formula with an error tending to zero as  $n$  grows and therefore  $p(n)$  being an integer, one could find the exact value.

If one looks at Ramanujan's first letter to Hardy, there is a statement there which has some relation to his later work on the partition function, namely about the coefficient of the reciprocal of a certain theta series (a power series with square exponents and alternating signs as coefficients). It gives the leading term in what he claims as an approximate expression for the coefficient. If one looks at that expression, one sees that this is the exact analogue of the leading term in the Rademacher formula for  $p(n)$  which shows that Ramanujan, in whatever way he had obtained this, had been led to the correct term of that expression.

In the work on the partition function, studying the paper it seems clear to me that it must have been, in a way, Hardy who did not fully trust Ramanujan's insight and intuition, when he chose the other form of the terms in their expression, for a purely technical reason, which one analyses as not very relevant. I think that if Hardy had trusted Ramanujan more, they should have inevitably ended with the Rademacher series. There is little doubt about that.

Littlewood and Hardy were primarily working with hard analysis and they did not have a strong feeling for modular forms and such things; the generating function for the partition function is essentially a modular form, particularly if one puts in an extra factor  $x^{-1/24}$  to the power series. This must have been something that came quite naturally to Ramanujan from the beginning. But to Littlewood, in this review, it seems as if it was an afterthought



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by a particular stroke of genius that happened later in the development. I find this completely misleading. Littlewood was not present when this happened; he was away from Cambridge during most of the years Ramanujan was there. Littlewood's memory also fails him when he refers to the statement that Ramanujan made in this letter to Hardy as referring to the partition function and not another but rather similarly built function.

At a later stage, when Hardy published the book *12 Lectures on Ramanujan*, Louis J Mordell reviewed this book and he questioned Hardy's assessment that Ramanujan was a man whose native talent was equal to that of Euler or Jacobi. Mordell questions whether the expression "native talent" has any real meaning and claims that one should judge a mathematician by what he has actually done, by which Mordell seems to mean, the theorems he has proved.

By the way, I should say Mordell clearly at no stage seems to have had access to or seen Ramanujan's *Notebooks*. Mordell's assessment seems quite wrong to me. I think that a felicitous but unproved conjecture may be of much more consequence for mathematics than the proof of many a respectable theorem.

Ramanujan's recognition of the multiplicative properties of the coefficients of modular forms that we now refer to as cusp forms and his conjectures formulated in this connection, and their later generalization, have come to play a more central role in the mathematics of today, serving as a kind of focus for the attention of quite a large group of the best mathematicians of our time. Other discoveries like the mock-theta functions are only in the very early stages of being understood and no one can yet assess their real importance. So the final verdict is certainly not in, and it may not be in for a long time, but the estimates of Ramanujan's stature in mathematics certainly have been growing over the years. There is no doubt about that.



It seems quite clear that Hardy's assessment of the degree of Ramanujan's gifts was quite correct, in spite of the great difference that there was between Hardy and Ramanujan in their particular mathematical leanings. One might speculate, although it may be somewhat futile, about what would have happened if Ramanujan had come in contact not with Hardy but with a great mathematician of more similar talents, someone who was more inclined in the algebraic directions, for instance, E Hecke in Germany. This might have perhaps proved much more beneficial and brought out new things in Ramanujan that did not come to fruition by his contact with Hardy. But Hardy deserves greatest credit for recognizing Ramanujan's originality and assisting him and his work in the best way he could.

I mentioned the assessment by Hardy of Ramanujan and I might quote from his preface to the *Collected Papers*. "Opinions may differ as to the importance of Ramanujan's work, the kind of standard by which it should be judged and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange. One gift it shows which no one can deny – profound and invincible originality." I think that is a very good phrase actually. "He would probably have been a greater mathematician if he could have been caught and tamed a little in his youth ..." and so forth, wrote Hardy. It is a true statement in one way and in some way it is not. I do not think that Hardy fully understood how the interest for Ramanujan's work would be growing, when he speaks of the influence which it is likely to have on the mathematics of the future. It seems rather clear that he underestimated that. Later developments have certainly shown him wrong on that point.

Hardy refers to Ramanujan also as one of the few romantic figures in the history of mathematics and also, in another place, as the one great romance of his life. Other romantic figures one may think of are, for instance, Galois or Abel. They died of course even younger but they did, although they had their

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difficulties, come from a somewhat more fortunate environment. One may say that none of them seems to have been quite so exclusively committed to mathematics, with no other visible interest in their life as Ramanujan was, and in this, I think, he truly stands completely alone among the mathematicians. At least from what I have read about him, there was no other real interest for him than mathematics.

It is also futile to speculate on what Ramanujan or, for that matter, Abel or Galois might have accomplished further, if their lives had not been cut short. The peak of a mathematician's work, I think, usually occurs between 30 and 40 years of age. One might have expected in all these cases that the added contribution might have equalled or outweighed their previous work. What is not futile perhaps is to consider what would have been lost if Ramanujan had not finally been given his chance. He is not the first exceptionally gifted person who has suffered rejection in a complexity inflexible and rigid educational system. Galois also failed in his examination. Abel was considered a very mediocre student, also in mathematics, until he got an exceptional mathematics teacher in school who did recognize his talents. Einstein had considerable difficulties in school. And in quite another field, one could mention Thomas Edison who was described by his teachers as having an addled brain.

One sometimes hears it said that true genius will always make its way and be recognized in the end. I do not think that is true.

There really must have been cases in the past where the native talent was finally completely thwarted by an inelastic system and teachers without sufficient understanding for the rare and unusual student for whom an exception should have been made. And I think this lack of consideration could continue usually with impunity because the world would never know what had been lost. The most important lesson that one could draw from Ramanujan's story about the educational system is that allowances should be made for the unusual and perhaps lopsidedly



gifted child with very strong interests in one direction, at all stages of the educational system.

There is also another thing which, I think, is rather important, and this is the school mathematics. I have talked with many others who became mathematicians, about the mathematics they learned in school. Most of them were not particularly inspired by it but started reading on their own, outside of school by some accident or the other, as I myself did. I think mathematics in the school definitely should be revised in such a way that it gives more of a sense of discovery and excitement. I think, in this, the teaching of mathematics often differs from the teaching of the other sciences which in the school is usually better carried out and does give a sense of discovery and excitement. And besides the schools, I think, it is also important for the development of possible future Ramanujans, that public libraries should stock a reasonable amount of mathematical books that could inspire and really interest someone who wants to find something outside of his school curriculum. This is one important thing that can be done in the future for making it easier for any future Ramanujan.

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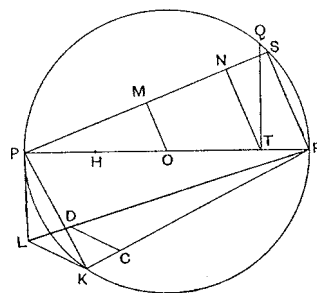
### Squaring The Circle, S Ramanujan



Let  $PQR$  be a circle with centre  $O$ , of which a diameter is  $PR$ . Bisect  $PO$  at  $H$  and let  $T$  be the point of trisection of  $OR$  nearer  $R$ . Draw  $TQ$  perpendicular to  $PR$  and place the chord  $RS = TQ$ .

Join  $PS$ , and draw  $OM$  and  $TN$  parallel to  $RS$ . Place a chord  $PK = PM$ , and draw the tangent  $PL = MN$ . Join  $RL$ ,  $RK$  and  $KL$ . Cut off  $RC = RH$ . Draw  $CD$  parallel to  $KL$ , meeting  $RL$  at  $D$ .

Then the square on  $RD$  will be equal to the circle  $PQR$  approximately.



*Journal of the Indian Mathematical Society, V, 1913, 132*