

# Classroom

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*In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.*

## What is a Hilbert Space?

When I was an undergraduate (aeons ago!), I overheard my seniors (exalted PhD students) discussing a mysterious object called a *Hilbert space*. It was many years later that I found out what it was, and then it occurred to me that stripped of all the jargon I could quite easily have absorbed this concept even as an undergraduate. Of course, undergraduates have come a long way since my days, and these days they know basic linear algebra and the theory of metric spaces. So my task of explaining the concept of a Hilbert space becomes infinitely simpler!

For simplicity we consider vector spaces over  $\mathbb{R}$ , the field of real numbers. Actually, to begin with, we need to know only the familiar space  $\mathbb{R}^n$ , the vector space of  $n$ -tuples of real numbers.

Apart from the algebraic operations of addition and scaling of vectors, we also have the notion of *dot product* of two vectors. If  $\underline{x} = (x_1, \dots, x_n)$  and  $\underline{y} = (y_1, \dots, y_n)$  are two vectors, the dot product  $\underline{x} \cdot \underline{y}$  is defined as the real number

$\sum_{i=1}^n x_i y_i$ . Notice that  $\underline{x} \cdot \underline{y} = \underline{y} \cdot \underline{x}$  and if  $\lambda_1, \lambda_2$  are scalars,  $(\lambda_1 \underline{x} + \lambda_2 \underline{y}) \cdot \underline{z} = \lambda_1 \underline{x} \cdot \underline{z} + \lambda_2 \underline{y} \cdot \underline{z}$ . This number can be used to calculate the angle  $\theta$  between two vectors via the formula

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$\cos \theta = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \|\underline{y}\|}$ , where  $\|\underline{x}\|$  denotes the *length* of the vector  $\underline{x}$  i.e.  $\|\underline{x}\| = \sqrt{x_1^2 + \dots + x_n^2} = (\underline{x} \cdot \underline{x})^{1/2}$ . Thus two vectors are perpendicular or *orthogonal* to each other if  $\underline{x} \cdot \underline{y} = 0$ . We can define the *distance* between two vectors  $\underline{x}$  and  $\underline{y}$  by  $d(\underline{x}, \underline{y}) = \|\underline{x} - \underline{y}\|$ . (See *Box 1*.) An important property of  $\mathbb{R}^n$  is *completeness* i.e. if  $\underline{x}^{(k)}$  is a Cauchy sequence of vectors (see *Box 1*), then there exists a *unique vector*  $\underline{x}$  such that  $\underline{x}^{(k)} \rightarrow \underline{x}$  as  $k \rightarrow \infty$ . As everyone knows, the dimension of  $\mathbb{R}^n$  as a real vector space is  $n$ . A basis  $\{\underline{e}_1, \dots, \underline{e}_n\}$  of  $\mathbb{R}^n$  is said to be an orthonormal basis if each  $\underline{e}_j$  has length 1 and  $\underline{e}_j$  is orthogonal to  $\underline{e}_k$  if  $j \neq k$ . Thus the 'standard' basis  $(1, 0, 0, \dots, 0), (0, 1, 0, \dots), \dots$  is an example of an orthonormal basis. (*Exercise*: In  $\mathbb{R}^3$ , if  $\underline{f}_1 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ , find  $\underline{f}_2$  and  $\underline{f}_3$ , so that  $\{\underline{f}_1, \underline{f}_2, \underline{f}_3\}$  is an orthonormal basis of  $\mathbb{R}^3$ . Note that we are *not* claiming that  $\underline{f}_2$  and  $\underline{f}_3$  are unique!) If all you are interested in is an  $n$ -dimensional real

**Box 1.**

A set  $X$  equipped with a map  $d : X \times X \rightarrow \mathbb{R}$  (called a *metric*) is called a *metric space* if the following axioms hold :

D1 :  $d(x, y) = d(y, x), \quad \forall x, y \in X$

D2 :  $d(x, y) \geq 0, \quad \forall x, y \in X$ , and  $d(x, y) = 0$  if and only if  $x = y$

D3 : (The triangle inequality)

$$d(x, y) + d(y, z) \geq d(x, z), \quad \forall x, y, z \in X$$

$d(x, y)$  is called the *distance* between  $x$  and  $y$ .

A sequence  $\{x^{(k)}\}$  in  $X$  is said to *converge* to  $x$  in  $X$  if and only if  $d(x^{(k)}, x) \rightarrow 0$  as  $k \rightarrow \infty$ .

A sequence  $\{x^{(n)}\}$  in  $X$  is said to be a *Cauchy sequence* in  $X$  if given  $\varepsilon > 0, \exists N(\varepsilon)$  such that  $\forall k, \ell \geq N(\varepsilon), d(x^{(k)}, x^{(\ell)}) \leq \varepsilon$ .

A metric space  $X$  is said to be *complete* if every *Cauchy sequence*  $\{x^{(k)}\}$  necessarily converges to a (unique)  $x$  in  $X$ .



Hilbert space, you don't have to go any further. Every real  $n$ -dimensional Hilbert space can be 'identified' (in a suitable sense) with  $\mathbb{R}^n$ .

What about infinite dimensional real Hilbert spaces? Let us denote the set of all real sequences  $\underline{x} = (x_1, x_2, \dots)$  with the added proviso that  $\sum x_i^2 < \infty$ , by  $\ell^2(\mathbb{Z}^+)$ . (This is quite a natural restriction since one wants the 'length' of a vector to be finite!)  $\ell^2(\mathbb{Z}^+)$  becomes a vector space over  $\mathbb{R}$  with the usual operations of addition of two vectors and the scaling of a vector by a real number. Unlike  $\mathbb{R}^n$ , a vector  $\underline{x}$  in this space may have infinitely (but countably) many non zero components. Once again define the dot product of two vectors  $\underline{x} = (x_1, x_2, \dots)$  and  $\underline{y} = (y_1, y_2, \dots)$  by  $\underline{x} \cdot \underline{y} = \sum_{i=1}^{\infty} x_i y_i$ . (This series converges absolutely. (Why?)) Thus,

as in the case of  $\mathbb{R}^n$ , we can talk about the angle between two vectors and also about two vectors being orthogonal. The length of  $\underline{x}$  is denoted by  $\|\underline{x}\|$  and is just the non negative number  $(\underline{x} \cdot \underline{x})^{1/2}$ . The distance  $d(\underline{x}, \underline{y})$  between two vectors  $\underline{x}$  and  $\underline{y}$  is defined to be  $\|\underline{x} - \underline{y}\|$ . Equipped with  $d(\cdot, \cdot)$ ,  $\ell^2(\mathbb{Z}^+)$  is a complete metric space (see Box 1). Note that the vectors  $\underline{e}_1 = (1, 0, 0, \dots)$ ,  $\underline{e}_2 = (0, 1, 0, \dots)$ ,  $\underline{e}_3 = (0, 0, 1, \dots)$ ,  $\dots$  are pairwise orthogonal, i.e.  $\underline{e}_j \cdot \underline{e}_k = 0$  if  $j \neq k$ . Also, any vector  $\underline{x} = (x_1, x_2, \dots)$  can be symbolically represented as  $\underline{x} = \sum_{j=1}^{\infty} x_j \underline{e}_j$ . More precisely, if  $\underline{x}^{(N)} = (x_1, x_2, \dots, x_N, 0, 0, \dots)$ ,

then  $\underline{x}^{(N)} = \sum_{j=1}^N x_j \underline{e}_j$ , and the  $\underline{x}^{(N)}$  form a Cauchy sequence in the space  $\ell^2(\mathbb{Z}^+)$  with  $\underline{x}^{(N)} \rightarrow \underline{x}$  as  $N \rightarrow \infty$ .

Any sequence of unit vectors  $\underline{f}_j$  in  $\ell^2(\mathbb{Z}^+)$  with  $\underline{f}_j \cdot \underline{f}_k = 0$  if  $j \neq k$ , and with the property that any  $\underline{x} \in \ell^2(\mathbb{Z}^+)$  can be symbolically expressed as  $\underline{x} = \sum_{j=1}^{\infty} \lambda_j \underline{f}_j$  (in the sense described above) is called an orthonormal basis. In that case, it can be shown that  $\lambda_j$  must be necessarily  $\underline{x} \cdot \underline{f}_j$ . Note that an orthonormal basis of  $\ell^2(\mathbb{Z}^+)$  is not an algebraic basis of  $\ell^2(\mathbb{Z}^+)$  considered as a vector space over  $\mathbb{R}$ . (Why not?) Another interesting fact is that, if  $W$  is the  $n$ -dimensional

space spanned by  $f_1, f_2, \dots, f_n$ , then for any given  $\underline{x}$ , the vector closest to  $\underline{x}$  in  $W$  is the unique vector  $\underline{x}^{(n)} = \sum_{j=1}^n (\underline{x} \cdot \underline{f}_j) \underline{f}_j$ . ( $\underline{x}^{(n)}$  is called the *projection* of  $\underline{x}$  onto the subspace  $W$ .) Once again, the length of  $\underline{x}$  is given by  $\|\underline{x}\|^2 = \sum_{j=1}^{\infty} (\underline{x} \cdot \underline{f}_j)^2$ .

Note that, if  $V_n$  is the linear subspace of  $\ell^2(\mathbb{Z}^+)$  consisting of those vectors (= sequences) which are zero after the  $n^{\text{th}}$ -stage i.e. of the form  $(x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$ , then  $V_n$  can be identified with  $\mathbb{R}^n$ . (Exercise; How?) Thus, all finite dimensional Hilbert spaces 'sit inside'  $\ell^2(\mathbb{Z}^+)$ ! Finally, just as a finite dimensional real Hilbert space of dimension  $n$  can be identified with  $\mathbb{R}^n$ , *any infinite dimensional 'separable' real Hilbert space can be identified (in a suitable manner) with  $\ell^2(\mathbb{Z}^+)$ .* (See Box 2 for a definition of the term *separable*. Most of the Hilbert spaces of interest to physicists and mathematicians are *separable*!)

Here is an exercise for the more ambitious student who already knows a little bit about Fourier series: Let  $L^2_{\text{per}}(\mathbb{R})$

**Box 2.**

A vector space  $V$  over  $\mathbb{R}$  is said to be an *inner product space* if it is equipped with a map  $\langle, \rangle: V \times V \rightarrow \mathbb{R}$ , satisfying the following axioms:

IP1 :  $\langle x, y \rangle = \langle y, x \rangle, \quad \forall x, y \in V$

IP2 :  $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle, \quad \forall x, y, z \in V, a, b \in \mathbb{R}$

IP3 :  $\langle x, x \rangle \geq 0$ , and  $\langle x, x \rangle = 0$  if and only if  $x = 0$

(Very often  $\langle x, y \rangle$  is denoted by  $x \cdot y$  and is referred to as the *dot product* of  $x$  and  $y$ .)

We define a *metric*  $d$  by defining the *distance*  $d(x, y)$  between  $x, y \in V$  by  $d(x, y) = \langle x - y, x - y \rangle^{1/2} = \|x - y\|$ . (For  $v \in V, \langle v, v \rangle^{1/2}$  is denoted  $\|v\|$ .)  $V$  becomes a *metric space* with this notion of distance. (See Box 1.)

If  $V$  is *complete* with respect to this metric, then  $V$  is called a *Hilbert space*.

A *Hilbert space* which contains a *countable* and *dense* subset is called *separable*.

be the space of (measurable) real valued  $2\pi$ -periodic square integrable functions (i.e.  $\int_0^{2\pi} (f(x))^2 dx < \infty$ ) on  $\mathbb{R}$ , where the 'dot product' of two periodic functions  $f$  and  $g$  is defined by  $\int_0^{2\pi} f(x)g(x)dx$ . Question: How will you 'identify' this space with  $\ell^2(\mathbb{Z}^+)$ ? (Hint : What do the Fourier coefficients of a function with respect to  $\sin kx, k \geq 1$ , and  $\cos kx, k \geq 0$ , have to do with an element of  $\ell^2(\mathbb{Z}^+)$ ?) If you are not familiar with this jargon, skip this paragraph. If you have some familiarity and are still unable to solve this exercise, pester your mathematics/physics teacher for help!

Having found out what a Hilbert space is, the natural question to ask is: What does one do with this object? Well, that is an entirely different story, or as they say, an entirely different ball game altogether! Another kind of object that you may have heard of is a *Banach space*. Just as a *Hilbert space* is a generalization of  $\mathbb{R}^n$ , one could say that a *Banach space* is a generalization of a *Hilbert space*. If you want to learn more about these fascinating subjects, we urge you to consult the excellent book of V S Sunder, *Functional Analysis*, Hindustan Book Agency, 1997.

Finally, a question that might still be nagging the historically minded reader is: Does the concept of a Hilbert space have anything to do with David Hilbert? The answer is both 'yes' and 'no'! Hilbert and many other mathematicians of his time were interested in certain concrete Hilbert spaces. In fact, many important results about Hilbert spaces can be traced back to the early work of Hilbert – but the axiomatization of a Hilbert space, as we know it today, is due to another great mathematician, John von Neuman.

**Here is Another Definition of a Hilbert Space!**

We quote from Constance Reid's biography of Courant:

The new building [Mathematics Institute of the University of Gottingen] – a three level T-shaped structure - provided everything the mathematicians had ever needed or wanted in their physical surroundings ... . The basement contained such requirements as a bicycle room, a book bindery, and a room for refreshments. The double stairs of the main entrance led to a spacious lobby, which today contains a bust of Hilbert and is known to students and faculty as "the Hilbert space"

