Chern and Total Curvature

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Shiing-Shen Chern[1] was one of the great mathematicians of the twentieth century, and a towering figure in the field of differential geometry, where one studies *curvatures* of spaces. In this article we give the reader a glimpse into differential geometry and the work of Chern.

1. Wind-vanes on Spheres

Let us begin with a few thought experiments. Consider a frictionless wind-vane in vacuum, as in *Figure* 1. Assume that the wind-vane is initially stationary. Then if we rotate the base of the wind-vane, the arrow does not move (as there is no friction).

Now assume that the wind-vane is placed flat on a plane. Let us move the base of the wind-vane smoothly along a curve in the plane. Then the wind-vane points in a constant direction, i.e., the direction of the wind-vane gives *parallel* vectors along the curve. This gives a physical meaning to vectors at different points being parallel.

[1] See *Box*1 by B Sury on the next page.

Something more interesting happens if we perform the same experiment on the surface of a sphere (see *Figure* 2). Suppose the wind-vane is initially placed at the point A which we take to be the north pole, and points in the direction of B on the equator. Let C be another point on the equator. We shall move the wind-vane along the longitude AB, then along the equator from B to C and finally back to A along a longitude.

Keywords
Gauss–Bonnet theorem, curvature, parallel transport, Euler characteristic, Chern class.

As we move along AB, the wind-vane is always pointing in the direction of motion. We can see this by symmetry – as the two sides of the longitude of a sphere look the same, the there is no reason for the wind-vane to turn

Box 1. S S Chern

Shiing-Shen Chern (26 October 1911 - 3 December 2004) was one of the greatest differential geometers of the 20th century. His work on characteristic classes in fibre spaces (which are now called Chern classes) turned out to be important not only in mathematics but also in mathematical physics. His proof of the Gauss-Bonnet formula gave him immense satisfaction. During an interview which appeared in the Notices of the *American Mathematical Society*, Vol. 47, 1998, he said "this is one of my best works as it solved an important, fundamental classical problem and the ideas were very new." Indeed, in that interview, he revealed that his work on characteristic classes "did not take me that much thought." In the 1940's, global differential geometry was just beginning when Chern entered the scene. It is largely due to Chern that the subject is a major one today.

Chern received his MS degree in 1934 from Peking, and his doctor of sciences degree from the University of Hamburg in 1936. He was a member of the Institute for Advanced Study at Princeton, from 1943 to 1945. During 1946-48 he was in China and was the acting director of the Institute of Mathematics at the Academia Sinica.

He returned to the United States in 1949 and taught at the University of Chicago and later at the University of California, Berkeley. Chern founded the now famous Mathematical Sciences Research Institute at Berkeley.

He won numerous honours throughout the world; he was awarded the US National Medal of Science in 1975, the Wolf Prize in 1983/84 and, in 1985, he was elected a Fellow of the Royal Society of London.

Chern was not only a towering figure in mathematics but was also a source of inspiration to several young mathematicians and physicists. He was very approachable and won loyalty from many adoring students; the 'Chern Visiting Professorship' in Berkeley was set up by one of Chern's students who had won a 22 million dollar lottery. The Nobel Laureate C-N Yang and the Fields Medalist S-T Yau were among the large number of Chern's students who made substantial contributions to mathematics and physics. The fact that he presided over the opening of the ICM 2002 in Peking alongside President Jiang Zemin bears testimony to his stature.

In the 1998 *Notices* interview from which we quoted above, he says, "I have no difficulty in mathematics, so when I do mathematics, I enjoy it. And therefore, I am always doing mathematics, because the other things I cannot do. I am retired now for many years and people ask me if I still do mathematics. And I think my answer is it's is the only thing I can do. There's nothing else I can do. And this has been true throughout my life."

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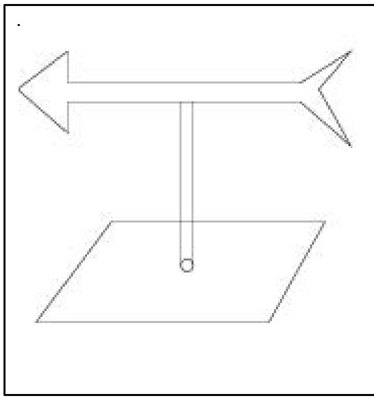

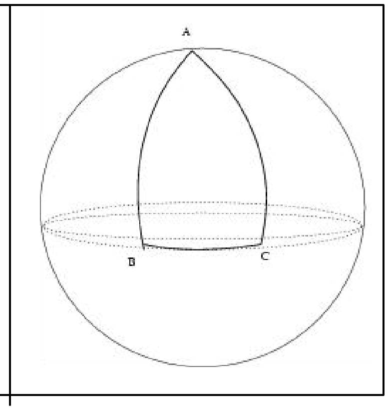

Figure 1 (left). The wind-vane.
Figure 2 (right).

right rather than left (or vice versa). When we reach B, the wind-vane is perpendicular to the equator, so that it is pointing due south. Now as we move along the equator to C (by symmetry) it continues to point south. Hence when we reach C, we are pointing along the longitude through C towards the south pole. Finally, as we move the wind-vane back along the longitude AC, it continues to point towards C.

Notice that something remarkable has happened. The wind-vane which was pointing towards B is now pointing towards C. Thus if we look at *parallel* vectors along the loop $ABCA$, we end up with a different vector from the one we had started with. Thus *parallel transport* around a loop on a sphere leads to a rotation, in technical language the *holonomy* of the parallel transport. This happens because the sphere is curved, and the amount by which we rotate depends on the *curvature*. By contrast, in the plane the holonomy is always the identity, i.e., each vector is taken to itself.

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2. Parallel Transport and Curvature

We shall now take a more geometric look at *parallel transport* and *curvature*. Suppose we have a curve α on a surface S from a point P to a point Q and a vector V at P that is tangent to S. Then we can *parallel transport* the vector V along α to get a parallel vector field along α. This corresponds to moving our wind-vane.

Let us see what properties such a parallel vector field should satisfy. Firstly, the vectors of the vector field have the same length. Further, the angle between two parallel vector fields is constant. Properties similar to this will hold for parallel transport in a more general setting. These properties amount to saying that the holonomy about a loop is a rotation.

The third property is more subtle. If we move in a straight line in the plane at constant speed, then the *velocity vector* clearly gives a parallel vector field along the line. We have a similar property for more general surfaces. The *shortest* curve on a surface S joining two points P and Q is a *geodesic*. If we move along a geodesic at constant speed, then the velocity vectors are parallel. This condition is called *torsion freeness* and will play a different role from the first two conditions in the general setting.

It turns out that these conditions essentially determine parallel transport. Observe that both these conditions are *intrinsic*, i.e. are determined by distances and angles on the surface and make no reference to the embedding of the surface in Euclidean space. This has an important consequence – we cannot make a map on a plane of a piece of a sphere without distorting distances[2]. This is because if we had such a map, then parallel vector fields on the sphere will map to parallel vector fields on the plane. So the holonomy should be the same in both cases. But we have seen that the holonomy is non-trivial for loops on the sphere, while it is the identity for loops on the plane.

[2] See also U Mukhopadhyay, Mercator and his Map, *Resonance*, Vol.10, No.3, pp.8-18, 2005.

The *shortest* curve on a surface *S* joining two points *P* and *Q* is a *geodesic*. If we move along a geodesic at constant speed, then the velocity vectors are parallel.

The *curvature* of a surface measures the holonomy. More precisely, let us take a small closed curve, say a parallelogram beginning and ending at a point P. We measure the angle by which parallel transport along the curve rotates vectors. We divide this by the area enclosed by the curve. As we take smaller and smaller curves, this ratio approaches a number which we call the *curvature* $\square(P)$ at P.

3. Total Curvature of a Sphere

We have seen that we cannot make a map from a piece of the sphere without distorting distances as its curvature is not zero. Suppose we smoothly deform the sphere. We can certainly make part of the sphere flat (i.e., make the curvature in this region zero). But intuition tells us we cannot make the curvature zero everywhere, we can make a region flat only at the cost of making some other region curved.

Indeed our intuition is right, and in fact there is a remarkable result, the Gauss–Bonnet theorem, which tells us that as we deform the sphere, the *total curvature* is unchanged! More precisely, we always have

$$\frac{1}{2\pi}\int_{S^2} \square = 2.$$

The Gauss–Bonnet theorem can be interpreted as relating the *local* geometric properties of the sphere to its *global* topological properties. Before going into this further, we make a digression into *topology*.

4. The Euler Characteristic

The Gauss–Bonnet theorem tells us that as we deform the sphere, the *total curvature* is unchanged! The theorem can be interpreted as relating the *local* geometric properties of the sphere to its *global* topological properties.

Water flowing on a sphere must be stationary at some point. This is not so for water flowing on a torus – the surface of a doughnut (or rubber tube). Formally, any smooth vector field on a sphere (the velocity of the water) has to be zero somewhere, but this is not so for a torus (see *Figure* 3).

The Poincaré–Hopf theorem says this and more. Suppose we have a smooth vector field on a surface S with finitely many zeroes. Then we can associate an index to each zero (similar to the multiplicity of a root of a polynomial). If the surface S is a sphere, then the *total index* is always 2 (remember the 2 in the Gauss–Bonnet

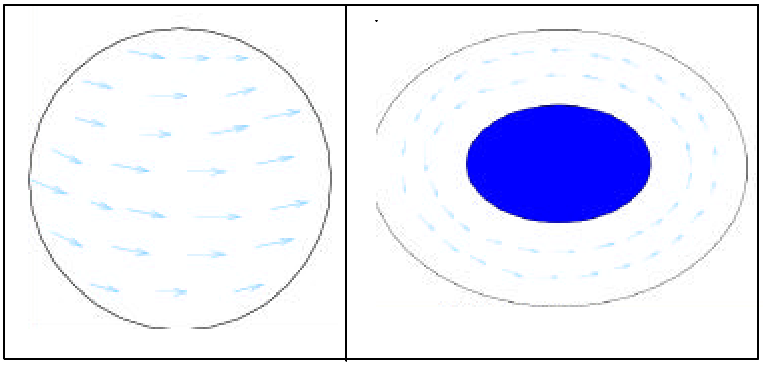

Figure 3.

theorem)! Similarly, in the case of a torus it is always zero. In general, for a surface S, the total index is always a fixed integer called *the Euler characteristic* $\chi(S)$ of the surface S.

This is a theorem in *topology* because it does not depend on the sphere being round – it is also true for the surface of an egg. Topological properties are those that are preserved by any smooth deformation (or any continuous deformation). The Euler characteristic is such a property – smoothly deforming S does not change $\chi(S)$.

Notice that it is easy to construct a vector field on a piece of a sphere that has no zeroes. In other words, we can make sure that water is not stationary anywhere in a small region. So the Poincaré–Hopf theorem is a *global* property of the sphere. Topological properties are the *global* properties of a space.

In higher dimensions, there are other numbers similar to the Euler characteristic associated to a space. The Euler characteristic is in fact the simplest of the *Chern numbers*, constructed by Chern. Other characteristic numbers were constructed independently by Steifel, Whitney and Pontrjagin.

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5. More on Total Curvature

Armed with the Euler characteristic, we now state the more general Gauss-Bonnet theorem. For a surface S, we have

$$\frac{1}{2\pi}\int \Box = \chi(S).$$

Observe that the integrand on the left hand side is *local* (can be measured on a small piece of the surface), *geometric* (depends on lengths and angles) and *real valued*. However on integrating we get the right hand side, which is *global*, *topological* and an *integer*!

Let us now turn to the higher dimensional situation. Here the notion of curvature is more complicated. Firstly the curvature is different in different directions. More precisely, if we take a small parallelogram near a point P, then the holonomy depends on the plane spanned by the parallelogram.

The holonomy about a loop still preserves lengths and angles, so is still an orthogonal transformation. In 2-dimensions the orthogonal transformations are just rotations, and so can be specified by a single number which is the angle by which we rotate. This is no longer true in general. For instance, in three dimensions, to specify a rotation we need to also specify its axis.

Nevertheless all these curvatures can be packaged together appropriately to form the *curvature tensor* (in the original approach due to Riemann) or the *curvature form* (in the approach of Elie Cartan). From the curvature form we can extract a real number – the Pfaffian, at each point of our space. The generalised Gauss-Bonnet theorem, due to Allendoerfer, Weil and Chern, says that, in even dimensions, the integral of the Pfaffian is the Euler characteristic.

Chern–Weil integrands, extracted from the curvature form give various *Chern numbers*. This constitutes *Chern–Weil theory*, whose proper setting is the *differential geometry of fibre spaces*.

More generally, the integrals of various other numbers, so called *Chern–Weil integrands*, extracted from the curvature form give various *Chern numbers*. This constitutes *Chern–Weil theory*, whose proper setting is the *differential geometry of fibre spaces*, to which we now turn. This was developed to a large extent by Chern.

Chern himself regarded the differential geometry of fibre spaces and his version of the Gauss–Bonnet theorem as two of his most important pieces of work.

6. Bundles, Connections and Curvatures

The differential geometry of fibre spaces comes about by placing parallel transport in its proper setting. The case we considered involved the parallel transport of *tangent vectors* of a surface around a loop, which satisfied two kinds of properties. The first kind of properties ensured that the holonomy was an orthogonal transformation. We also had the condition of *torsion freeness*. We shall generalise all this.

Suppose S is a smooth surface contained in the usual 3-dimensional Euclidean space. At each point P in S, we can consider the tangent space to S at P, which is a vector space. The totality of these tangent spaces constitutes the *tangent bundle*.

There are other ways of associating vector spaces to each point of a space. For instance, for each P in S we can consider the space of vectors orthogonal to S at P. This is a one-dimensional space, and the collection of all such vector spaces forms the *normal bundle*.

Properties of the type – that lengths of parallel vector fields and angles between parallel vector fields are preserved – make sense for the normal bundle. However *torsion freeness*, i.e., that the velocity vector along a geodesic is parallel, no longer makes any sense as the velocity vector is not a normal vector.

The tangent bundle and the normal bundle are examples of *vector bundles*, which associated a vector space to each point of a vector space in a smooth way. The vector space associated to a point is called the *fibre*. An interesting example for us is a surface S embedded in 4-dimensional Euclidean space. In this case the normal bundle has 2-dimensional fibres.

Parallel transport for a vector bundle (including the tangent bundle) is the line integral of a quantity called the *connection*. We have seen that for the tangent space, parallel transport, hence the connection, is determined by two kinds of properties. Let us consider next the case of the normal bundle.

Properties of the first type – that lengths of parallel vector fields and angles between parallel vector fields are preserved – continue to make sense for the normal bundle. However *torsion freeness*, i.e., that the velocity vector along a geodesic is parallel, no longer makes any sense as the velocity vector is not a normal vector.

If we require the first type of properties for parallel vector fields, then we can find connections satisfying these. But such connections are no longer unique. Nevertheless we can pick such a connection and study differential geometry using the connection. Observe that we can still speak of the holonomy of a connection.

In general for a vector bundle we allow connections with the same properties of the first type. In the above cases these properties amounted to requiring that the holonomy is an orthogonal transformation of the fibre. Technically we say in this case that the *Gauge group* is the group of fibre-wise orthogonal transformations. We may require instead, for instance, that the Gauge group is the group of fibre-wise unitary transformations, or all fibre-wise complex linear transformations.

For any connection on a vector bundle, we can define the curvature form. From this we extract *Chern–Weil integrands*. The integrals of these are topological quantities associated to the vector bundle - its *Chern numbers*. All this is in the setting *differential geometry of fibre spaces*, developed largely by Chern.

We have thus seen a glimpse of Chern–Weil theory in its natural setting, where for any bundle and any connection on the bundle, the integrals of the *Chern-Weil integrands* give *Chern numbers* which are topological quantities depending on the bundle and *not* on the connection. Nevertheless, it is in many ways useful to have a special connection on a bundle – for instance we then

Suggested Reading

[1] Shiing-Shen Chern, A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds, *Ann.of Math.*, Vol.45, pp.747-752, 1944.

[2] Andrew Pressley, *Elementary Differential Geometry*, Springer-Verlag, 2001.

[3] John A Thorpe, *Elementary Topics in Differential Geometry*, Springer-Verlag, 1979.

[4] R O Wells Jr., *Differential Analysis on Complex Manifolds*, Springer-Verlag, 1980.

can talk of the curvature of the bundle rather than that of a particular connection on it.

We saw that on the tangent bundle, there is a unique connection (called the *Levi–Civita* connection) which is torsion-free. There is another situation where there is a unique connection satisfying appropriate properties. This is when we have a Hermitian holomorphic vector bundle on a complex space. This unique connection is the *Chern connection*!

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High sophie-stication

Sophie Germain is known now as a French mathematician who proved special cases of Fermat's last theorem : she proved that if p is a prime such that 2p + 1 is also a prime, then any solution to Fermat's equation $X^p + Y^p = Z^p$ must be so that p divides one of X; Y; Z. What may not be as well known is that since only men were allowed to study at the Ecole Polytechnique in Paris at that time (late 18th century), she had to take the identity of a former male student Monsieur Le Blanc. Sophie Germain could not attend classes at the Ecole Polytechnique but under her assumed identity, she could obtain lecture notes and problem sets intended for Le Blanc. She used to submit answers to the problems under this pseudonym. To the course instructor Lagrange, Germain's solutions indicated such a remarkable transformation in a student who had previously lacked any mathematical skill that Lagrange asked to meet the student. Germain was forced to reveal her identity and thus Lagrange became her mentor. Even when Germain wrote to Gauss, she used her pseudonym as she feared that the great man would not take her seriously because of her gender. Germain's contribution to Fermat's last theorem would have been forever attributed to the mysterious Monsieur Le Blanc were it not for Napoleon. In 1806, Napoleon was invading Prussia and the French army was storming through one German city after another. Germain, fearing for the life of Gauss, sent a message to her friend, General Pernety, asking that he guarantee Gauss's safety. The general took special care of Gauss, explaining to him that he owed his life to Mademoiselle Germain. Gauss was grateful but surprised, for he had never heard of Sophie Germain. In Germain's next letter to Gauss she reluctantly revealed her true identity.

Sophie Germain also wrote an important paper on the theory of elasticity titled 'Memoir on the Vibrations of Elastic Plates'.

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