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Minimal classical communication and measurement complexity for quantum information splitting of a two-qubit state

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Abstract. We investigate the usefulness of the highly entangled five-partite cluster and Brown states for the quantum information splitting (QIS) of a special kind of two-qubit state using remote state preparation. In our schemes, the information that is to be shared is known to the sender. We show that, QIS can be accomplished with just two classical bits, as opposed to four classical bits, when the information that is to be shared is unknown to the sender. The present algorithm, demonstrated through the cluster and Brown states is deterministic as compared to the previous works in which it was probabilistic.

Keywords. Entanglement; teleportation; secret sharing; information splitting.

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1. Introduction

Entanglement is the most striking, as well as counter-intuitive feature of quantum mechanics. It has proved to be useful in a number of practical applications in the field of communication and cryptography [1]. It is understood only in the case of two particles. The multiparticle scenario is not well-characterized, owing to the increase in complexity with the number of qubits as there are numerous ways in which they can be entangled [2]. Entangled states are used as communication resources in teleportation, secret sharing and superdense coding. Quantum teleportation is a technique for transfer of information between parties, using a distributed entangled state and a classical communication channel.

Bennett *et al* [3], proposed the first scheme for the teleportation of an unknown single qubit state $\alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. In their scheme, Alice the sender combines the unknown qubit state with the EPR pair and performs a Bell measurement, and conveys the outcome of her measurement to Bob the receiver, via two classical bits of information. Based on the information sent by Alice, Bob performs appropriate unitary transformations on his particles and obtains the unknown qubit information. In this process, Alice and Bob get disentangled. Hence, the resources required to achieve this task, are one ebit of entanglement and two classical bits of information. This has been achieved experimentally in different quantum systems [4–6]. Recently, attention has turned towards the teleportation of an arbitrary two-qubit state [7–10] given by

$$|\psi_2\rangle = \alpha|00\rangle + \gamma|10\rangle + \beta|01\rangle + \delta|11\rangle, \tag{1}$$

where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. Two-qubit teleportation has been achieved using four- and five-partite genuinely entangled states consuming two ebits and four classical bits. This has been experimentally realized using pairs of Bell states [11]. Recently, the effect of decoherence on the standard teleportation protocol has been studied [12].

Quantum information splitting or secret sharing of quantum information, is the splitting up of quantum information between various parties, such that unless each of them cooperates, the receiver cannot obtain the desired information. Hillery *et al* [13] have demonstrated the sharing of an unknown single qubit of information among three parties using the three- and four-particle GHZ states. In their scheme, Alice, Bob and Charlie initially share a three-qubit GHZ state possessing one qubit each. Alice has an unknown qubit $\alpha|0\rangle + \beta|1\rangle$, which she wants Bob and Charlie to share. So, Alice combines the unknown qubit with her part of the entangled state, performs a Bell measurement and conveys the outcome of her measurement to Charlie via two classical bits. Now, the Bob–Charlie system evolves into a state given by $U_x(\alpha|00\rangle + \beta|11\rangle)_{23}$, where U_x is a unitary operator. Hence, the unknown single qubit information is locked between Bob and Charlie in such a way that neither of them can obtain the unknown qubit completely, by local operations on their own qubits. Now, Bob can perform a single qubit measurement in the basis $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ on his qubit and convey the outcome of his measurement to Charlie via one classical bit. Having known the outcomes of both their measurements, Charlie can obtain the unknown qubit by performing a suitable unitary operation. Later, schemes have been devised for the QIS of an unknown single qubit using a three-qubit asymmetric W state, which has been realized in ion trap systems [14]. QIS of a single qubit state was also experimentally observed by making use of single photon sources [15].

Remote state preparation has the same goal as that of teleportation, except that the sender knows the state that is to be teleported. It was shown by Pati [16] that when a qubit is chosen from an equatorial or real circles on a Bloch sphere, it can be remotely prepared using one cbit of information and one ebit of entanglement between Alice and Bob. The trade-off between the classical communication required and the amount of information known about the initial state was investigated in [17]. This gives us motivation to investigate remote state preparation protocols which will simultaneously help in achieving QIS.

Recently, QIS of an arbitrary two-qubit state was proposed by two of the present authors, utilizing the five-particle Brown and cluster states [9,18]. The procedure utilizes four cbits as information resource. In all these schemes, the state that is to be shared is unknown to the sender. However, in many practical scenarios, the

Table 1. The outcome of the measurement performed by Alice and the state obtained by Bob and Charlie using the four-qubit cluster state.

Outcome of the measurement	State obtained
$\frac{1}{2}(\alpha^*\|00\rangle + \beta^*\|01\rangle + \gamma^*\|10\rangle + \delta^*\|11\rangle)$	$\alpha\|00\rangle + \beta\|10\rangle + \gamma\|01\rangle - \delta\|11\rangle$
$\frac{1}{2}(\alpha^*\|00\rangle + \beta^*\|01\rangle - \gamma^*\|10\rangle - \delta^*\|11\rangle)$	$\alpha\|00\rangle + \beta\|10\rangle - \gamma\|01\rangle + \delta\|11\rangle$
$\frac{1}{2}(\alpha^*\|00\rangle - \beta^*\|01\rangle - \gamma^*\|10\rangle + \delta^*\|11\rangle)$	$\alpha\|00\rangle - \beta\|10\rangle - \gamma\|01\rangle - \delta\|11\rangle$
$\frac{1}{2}(\alpha^*\|00\rangle - \beta^*\|01\rangle + \gamma^*\|10\rangle - \delta^*\|11\rangle)$	$\alpha\|00\rangle - \beta\|10\rangle + \gamma\|01\rangle + \delta\|11\rangle$

information that is to be shared may be known to the sender. In such cases, the classical information resources can be greatly reduced and need not be wasted. In this paper, we extend these results to a two-qubit state by investigating the five-particle cluster [19] and Brown [20] states as entangled resources. Let Alice possess the two-qubit information, given in eq. (1) with $\alpha = \delta = \frac{1}{2}$, $\beta = \gamma = \frac{1}{2}e^{i\phi}$, which she wants Bob and Charlie to share. To illustrate the difference between pure remote state preparation and remote state preparation in conjunction with QIS, we start with the protocol for the remote state preparation of $|\psi_2\rangle$ using a four-qubit cluster state;

$$|C_4\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle), \tag{2}$$

as an entangled resource. In this protocol, Alice and Bob possess the first and the last two qubits of $|C_4\rangle$ respectively. Alice also possesses $|\psi_2\rangle$ which she wants to teleport to Bob. Since Alice is aware of the information that is to be sent, she can perform a two-particle measurement in the basis involving the coefficients of the input state. The outcome of the measurement performed by Alice and the state obtained by Bob are shown in table 1. Now Bob can obtain $|\psi_2\rangle$ by performing an appropriate controlled phase shift operation between the two qubits to obtain $|\psi_2\rangle$. This completes the remote state preparation of $|\psi_2\rangle$ using $|C_4\rangle$. We now proceed to investigate the usefulness of the five-qubit cluster and the Brown states for the remote state preparation of $|\psi_2\rangle$ using QIS.

2. Cluster state for QIS of a two-qubit state

The five-particle cluster state is given by

$$|C_5\rangle = \frac{1}{2}(|00000\rangle + |00111\rangle + |11101\rangle + |11010\rangle). \tag{3}$$

There are two ebits of entanglement between pairs 15|234, which makes $|C_5\rangle$ a useful resource for quantum communication. It has been shown that, this state can be used for the teleportation of single and two-qubit states [21]. Moreover, it is known that, the superdense coding capacity of this state reaches the Holevo bound allowing five cbits to be sent using only three qubits [21]. Recently, it was found that $|C_5\rangle$ could be used for QIS of an arbitrary two-qubit state by utilizing

Table 2. The outcome of the measurement performed by Alice and the state obtained by Bob and Charlie using the five-qubit cluster state.

Outcome of the measurement	State obtained
$\frac{1}{2}(\alpha^*\|00\rangle + \beta^*\|01\rangle + \gamma^*\|10\rangle + \delta^*\|11\rangle)$	$\alpha\|000\rangle + \beta\|011\rangle + \gamma\|101\rangle + \delta\|110\rangle$
$\frac{1}{2}(\alpha^*\|00\rangle + \beta^*\|01\rangle - \gamma^*\|10\rangle - \delta^*\|11\rangle)$	$\alpha\|000\rangle + \beta\|011\rangle - \gamma\|101\rangle - \delta\|110\rangle$
$\frac{1}{2}(\alpha^*\|00\rangle - \beta^*\|01\rangle - \gamma^*\|10\rangle + \delta^*\|11\rangle)$	$\alpha\|000\rangle - \beta\|011\rangle - \gamma\|101\rangle + \delta\|110\rangle$
$\frac{1}{2}(\alpha^*\|00\rangle - \beta^*\|01\rangle + \gamma^*\|10\rangle - \delta^*\|11\rangle)$	$\alpha\|000\rangle - \beta\|011\rangle + \gamma\|101\rangle - \delta\|110\rangle$

two ebits and four cbits [10]. In this section, we show that the classical resource is reduced by half, provided the information that is to be shared belongs to a special case of a two-qubit state.

Let Alice possess qubits 1 and 5, and let Bob and Charlie possess the remaining qubits. Since Alice is aware of the information that is to be sent, she can perform a two-particle measurement in the basis involving the coefficients of the input state. The outcome of the measurement performed by Alice and the state obtained by Bob and Charlie are shown in table 2. Neither Bob nor Charlie can obtain the information through local operations on their own qubits. Alice encodes the outcome of her measurement in classical bits and sends it to Charlie.

Now, Bob can perform a measurement in the basis $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and convey its outcome to Charlie via one classical bit. For instance, if the Bob–Charlie system collapses to the first state in the table, after Bob performs the measurement, then Charlie's system collapses to $\frac{1}{2}(|00\rangle + \mathrm{e}^{i\phi}|11\rangle + \mathrm{e}^{i\phi}|01\rangle + |10\rangle)$ or $\frac{1}{2}(|00\rangle + \mathrm{e}^{i\phi}|11\rangle - \mathrm{e}^{i\phi}|01\rangle - |10\rangle)$. While in the first case, Charlie needs to apply a *CNOT* gate between his qubits to obtain $|\psi_2\rangle$, in the second case Charlie needs to apply a *CNOT* gate between his qubits followed by a $(I \otimes \sigma_3)$ operation to obtain $|\psi_2\rangle$. In the original scheme involving the cluster state for QIS of an arbitrary two-qubit state, Alice performs a four-particle measurement and conveys the outcome of her measurement to Charlie via four cbits of information. In comparison, the present scheme uses only two cbits of information, provided the two-qubit state is chosen of the type discussed earlier. Hence for these special states this procedure reduces measurement complexity and the consumption of classical information by two cbits.

3. Brown state for QIS of a two-qubit state

The Brown state is given by

$$|\psi_5\rangle = \frac{1}{2}(|001\rangle|\phi_-\rangle + |010\rangle|\psi_-\rangle + |100\rangle|\phi_+\rangle + |111\rangle|\psi_+\rangle), \quad (4)$$

where $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are Bell states. These states were obtained by an extensive numerical optimization procedure [20]. It is highly entangled and this has been checked by yet another numerical search procedure [22]. This state exhibits genuine multi-partite entanglement according to many measures of entanglement [9]. Like $|C_5\rangle$, the superdense coding capacity

Table 3. The outcome of the measurement performed by Alice and the state obtained by Bob and Charlie using the Brown state.

Outcome of the measurement	State obtained
$\frac{1}{2}(\alpha^*\|00\rangle + \beta^*\|01\rangle + \gamma^*\|10\rangle + \delta^*\|11\rangle)$	$\alpha\|\eta_1\rangle + \beta\|\eta_2\rangle + \gamma\|\eta_3\rangle + \delta\|\eta_4\rangle$
$\frac{1}{2}(\alpha^*\|00\rangle + \beta^*\|01\rangle - \gamma^*\|10\rangle - \delta^*\|11\rangle)$	$\alpha\|\eta_1\rangle + \beta\|\eta_2\rangle - \gamma\|\eta_3\rangle - \delta\|\eta_4\rangle$
$\frac{1}{2}(\alpha^*\|00\rangle - \beta^*\|01\rangle - \gamma^*\|10\rangle + \delta^*\|11\rangle)$	$\alpha\|\eta_1\rangle - \beta\|\eta_2\rangle - \gamma\|\eta_3\rangle + \delta\|\eta_4\rangle$
$\frac{1}{2}(\alpha\|00\rangle - \beta\|01\rangle + \gamma^*\|10\rangle - \delta^*\|11\rangle)$	$\alpha\|\eta_1\rangle - \beta\|\eta_2\rangle + \gamma\|\eta_3\rangle - \delta\|\eta_4\rangle$

of this state reaches the maximum, allowing five classical bits to be transmitted by only three quantum bits [21]. Recently, it was shown that, this state can be used for the QIS of single and two-qubit states [10]. In this section, we show that the classical resource is reduced by half, if the information that is to be shared belongs to the special class of the two-qubit state discussed earlier.

Let Alice have the first two particles, Bob have the third one and Charlie the last two particles. In this scheme also Alice performs measurement in an appropriate basis. The outcome of the measurement performed by Alice and the Bob–Charlie system is shown in table 3. The states $|\eta_i\rangle$ used in table 3 are

$$|\eta_1\rangle = \frac{1}{2}(|101\rangle - |110\rangle), \tag{5}$$

$$|\eta_2\rangle = \frac{1}{2}(|000\rangle - |011\rangle), \tag{6}$$

$$|\eta_3\rangle = \frac{1}{2}(|001\rangle + |010\rangle), \tag{7}$$

$$|\eta_4\rangle = \frac{1}{2}(|100\rangle + |111\rangle). \tag{8}$$

As in the previous case, even here neither Bob nor Charlie can obtain the information through local operations on their own qubits. Alice encodes the outcome of her measurement, performed with equal probability into classical bits and sends it to Charlie.

Now, Bob can perform a measurement in the basis $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and convey its outcome to Charlie via one classical bit. For instance, if the Bob–Charlie system collapses to the first state in table 3, after Bob performs the measurement, Charlie's system collapses to $\frac{1}{2}(\pm|\phi_-\rangle + \mathrm{e}^{i\phi}|\psi_-\rangle + \mathrm{e}^{i\phi}|\phi_+\rangle \pm |\psi_+\rangle)$. Charlie can perform the unitary operation $(\pm|11\rangle\langle\psi_+| + |10\rangle\langle\phi_+| \pm (|00\rangle\langle\phi_-| + |01\rangle\langle\psi_-|))$ and obtain $|\psi_2\rangle$. Hence, the protocol succeeds.

4. Conclusion

In conclusion, we have shown that for the QIS of a two-qubit state, the classical information resource is reduced by half, provided the two-qubit state is of a particular type. Hence, one need not waste classical information resource, when some information is known about the initial state. Interestingly, the present algorithm is deterministic, which was not possible in the protocol given in [23]. If one would implement the protocol given in ref. [23] twice, then one would need six qubits. However, we achieve this by entangling only five qubits. In future, it is worth checking, if classical or quantum information resources in other protocols can be reduced when some information about the initial state is known to the sender. Generalization to the QIS of real or equatorial N qubit states using $2N$ and a $(2N+1)$ qubit states as entangled resources need to be explored. The possibility of reducing the measurement complexity and the classical information for QIS using continuous variables also needs to be investigated. The methods introduced here is also a way of producing entangled states and encoding a qubit into a higher-dimensional space. This will be investigated shortly.

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