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Classical and quantum mechanics of complex Hamiltonian systems: An extended complex phase space approach

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Abstract. Certain aspects of classical and quantum mechanics of complex Hamiltonian systems in one dimension investigated within the framework of an extended complex phase space approach, characterized by the transformation $x = x_1 + ip_2$, $p = p_1 + ix_2$, are revisited. It is argued that Carl Bender inducted $\mathcal{PT}$ symmetry in the studies of complex power potentials as a particular case of the present general framework in which two additional degrees of freedom are produced by extending each coordinate and momentum into complex planes. With a view to account for the subjective component of physical reality inherent in the collected data, e.g., using a Chevreul (hand-held) pendulum, a generalization of the Hamilton's principle of least action is suggested.

Keywords. Complexification methods; complex Lagrangian; Chevreul pendulum; generalized Hamilton's principle.

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1. Introduction

1.1 *Motivation behind the study of complex Hamiltonians*

It is believed that the consciousness of a Being in general and that of a human Being in particular is responsible for the creation of space, time and geometry in Nature (the so-called spatio-temporal Cartesian grid (STCG)). Also, physical theories and their viability become questionable as and when a scientist thinks of applying them to explain data that do not lie within this grid; for example, the data taken in the behavioural domain involving living Beings. Besides the conventional complexity that arises in treating the space and time variables on the same footing in the relativity theory, physical theories by and large have been formulated on a real STCG. As a matter of fact the explanations of most of the data in the past did not require any kind of complexity either of variables, their derived variants, or their functions appearing in the theory. In spite of all this, the interpretation of the quantum wave function demanded the complexity and in that too the complexity appears at the conceptual level by keeping not only the arguments of the wave function but also the end-product of the observables (i.e., eigenvalues) as real. Here, perhaps one wants to account for 'something' that is beyond the reality described by STCG, i.e., a kind of missing component of physical reality (subjectivity component) involved in the quantum measurement problem. Can one say in general that the complexification of any form (say, of dynamical variables or of their functions within the framework of STCG or of the parameters of the system) is going to account for the subjective component of physical reality involved in the data? A search for a possible answer to this question will be made in this paper.

It is only during the last decade or so that the study of complex Hamiltonian systems has become of considerable theoretical interest [1] in both classical and quantum domains without actually noticing much of its necessity on the experimental front. In any case, the theories developed or going to be developed with complex physical variables or with complex Hamiltonians may turn out to be useful in more ways than one in future. Besides providing a mathematical beauty of the theory in the form of its generalization in its own right, such complexification will perhaps account for (i) the new features, if observed, in the data, and/or (ii) the subjective component of physical reality, if traced or found to exist in the data collected by a conscious observer (for example, as is the case in the quantum measurement problem or the case of a Chevreul (hand-held) pendulum in the classical domain). It is well-known that physical laws work well for a closed system and their viability is often questioned when the system is allowed to become open. Another point needs to be explored and perhaps is too early to speculate, is whether the complexification of a dynamical system is a step in the direction of accounting for the openness of a system if it exists.

1.2 *Methods of complexifying a Hamiltonian*

As such a Hamiltonian is a physically realizable mathematical construct expressed generally as the sum of kinetic and potential energy terms, and in one dimension it has the form

$$H(x,p) = \frac{p^2}{2m} + V(x), \tag{1}$$

where, like the mass m in the first term, the potential energy $V(x)$ can also contain certain parameters. Note that in (1), for a physical system while the form of the first term is reserved, the second term, $V(x)$, can have different functional forms depending upon the system under study. Let us consider different possible ways in which a complex form of H can be produced out of the real H in (1).

Method C1: H can be made complex by assuming only the potential function $V(x)$ as complex, say $V(x) = V_r(x) + iV_i(x)$. The simplest choice to this case is the complex square-well potential, namely $V(x) = V_0 + iW_0$, used by Feshbach *et al* in the optical model of nucleus about 60 years ago [2].

Method C2: H can be made complex by converting the real two-dimensional phase space (x–p plane) into a complex x–p plane (z-plane). If one defines

$$z = p + iw_0 x; \quad z^* = p - iw_0 x, \tag{2}$$

with w_0 as a real parameter and thus, $H(x,p) \to H(z,z^*)$. Then, such a complexification is found [3] to work successfully for harmonic oscillator potential or in the second quantized version of field theory through creation and annihilation operators. Another generalized version of (2) in the form

$$u = \frac{1}{\sqrt{2}}\left(\frac{x}{b} + i\frac{p}{c}\right), \quad v = \frac{1}{\sqrt{2}}\left(\frac{x}{b} - i\frac{p}{c}\right) \tag{3}$$

with x and p as real and b and c as complex, has also been used in some physical problems. In this case u, u^*, v and v^* become new degrees of freedom but again by complexifying the parameters, as in Method C3 below.

Method C3: H in (1) can be made complex by just considering the parameters in $V(x)$ as complex, including the mass parameter m as well; however after retaining x and p as real.

Method C4: $H(x,p)$ can be made complex by considering each physical variable x and p as complex, i.e., by extending the real two-dimensional phase space (x–p plane) to a complex space with four degrees of freedom. This is possible by writing x and p in either of the ways, namely

$$x = x_1 + ix_2; \quad p = p_1 + ip_2, \tag{4}$$

or

$$x = x_1 + ip_2; \quad p = p_1 + ix_2. \tag{5}$$

While the transformation (4) is used by Rao *et al* [4] in their studies of ion-acoustic waves in plasma and more recently by Yang [5] in developing a complex mechanics of which conventional classical and quantum mechanics may appear as special cases, the transformation (5) is used by Xavier and Aguiar [6] in their coherent state studies and by us [7–11] in studying certain aspects of classical [7,8] and quantum [9–11] mechanics of non-Hermitian systems. In fact (x_1, p_1) and (x_2, p_2) in (5) separately turn out [7] to be the canonical pairs. In what follows, the use of transformations (4) and (5) will be termed respectively as extended complex phase space approach (ECPSA) and extended complex configuration space approach (ECCSA).

Method C5: The most general way to produce complex H from the real H in (1) is by combining Methods C3 and C4, i.e., by considering both parameters and physical variables as complex. In this case, however, the extraction of the physics content from the mathematical construct (1) becomes a difficult task.

With regard to these methods of complexification the following remarks are in order: (1) Note that the form of the potential function $V(x)$ is responsible for the nature of trajectory of the particle whereas the finer details of this trajectory are taken care of by the parameters appearing in it. Therefore, any complexification of parameters cannot lead to a basic change in the nature of trajectory except for allowing some variations around its main track. In some sense the parametric complexification of H affects the geometrical properties but not the geometry of the space-time structure in Nature. On the other hand, the type of complexification inducted by (5) (or for that matter by (4)) brings in two additional (fictitious) variables x_2 and p_2 in hyperspace which may or may not have any link with the conventional STCG. They however follow the same rules of the game as the physical variables x_1 and p_1. Thus, while the variables x_1, p_1 and their functions will account for the physical reality in Nature, the variables x_2, p_2 and their functions will account for a reality that is beyond physical, i.e., for the spurious effects highlighted in §1.1.

(2) An important question which can naively be asked is: 'why one should use the same iota 'i' for the complexification of a number system and of the transformations on it'. In principle one should use different iotas, say i for a complex number system and I for a complex mapping on it. We shall return to some of these points in §4. However, the roots of the answer to this question lie in using the Cauchy–Riemann conditions for the analyticity of a function of complex variable. In fact it is a pre-requisite for the description of physical reality but may not be for the reality inducted through the transformation (5) (see Remark 1 above). As a result it is worthwhile to relax these conditions and see as to what happens to the resulting mathematical constructs after that (see §4).

(3) It can be argued that some of the approaches followed recently [1,12,13] to study the real eigenvalue spectra of non-Hermitian Hamiltonians are based either on the parametric complexification of H or are the particular cases of Method C4. For example, for real x and p, under combined $\mathcal{PT}$ symmetry one requires that $(x, p, i) \to (-x, p, -i)$ and the same gets translated into $(x_1, p_1, x_2, p_2, i) \to (-x_1, p_1, -x_2, p_2, -i)$ in Method C4, provided the newly introduced variables x_2, p_2 are designated as coordinate and momentum in the extended complex phase space. With this particular choice, the complex H in (1) becomes $\mathcal{PT}$-symmetric; otherwise the transformations (4) and (5) in general describe extensions of the real configuration and phase spaces.

2. Results in the classical and quantum domains using ECPSA

In the classical domain, using (5) the construction of exact complex invariants for a large number of complex Hamiltonian systems is carried out [8]. Also, the integrability of the two equivalent real Hamiltonian systems H_1 and H_2 in two dimensions, obtained by writing $H(x, p)$ as

$$H(x, p) = H_1(x_1, p_1, x_2, p_2) + iH_2(x_1, p_1, x_2, p_2), \tag{6}$$

is investigated [7]. For this purpose, one writes the time-dependent complex dynamical invariant $I(x, p, t)$ of the system as

$$I(x, p, t) = I_1(x_1, p_1, x_2, p_2; t) + iI_2(x_1, p_1, x_2, p_2; t), \tag{7}$$

and demands that

$$(\mathrm{d}I/\mathrm{d}t) = (\partial I/\partial t) + [I, H] = 0, \tag{8}$$

where $[\cdot,\cdot]$ is the Poisson bracket. Some of the complex Hamiltonians for which the complex dynamical invariants constructed are [8]: harmonic oscillator, $H = (p^2/2) + (w^2x^2/2)$; shifted harmonic oscillator on a complex plane, $H = p^2 + x^2 + ix$; $\mathcal{PT}$-symmetric $H = p^2 + \delta(ix) + \delta_2(ix)^2 + \delta_3(ix)^3$; time-dependent harmonic oscillator, $H = (p^2/2) + (w^2(t)x^2/2)$; $\mathcal{PT}$-symmetric $H = (p^2/2) + ix + ix^3$; non-$\mathcal{PT}$-symmetric $H = (p^2/2) + x + ix^3$, and quartic oscillator $H = (p^2/2) + (\delta_1 x^2/2) + (\delta_2 x^4/4)$.

In the quantum domain, we solve the eigenvalue equation

$$\hat{H}(x,p)\Psi(x) = E\Psi(x), \tag{9}$$

where $\hat{H}(x,p)$ is the operator version of $H(x,p)$ in (1) with $p \to -i\partial/\partial x$, namely

$$\hat{H}(x,p) = -(\partial^2/2\partial x^2) + V(x), \tag{10}$$

for the complex potential $V(x)$ and using $\hbar = m = 1$. Next, we write

$$\psi(x) = \psi_r(x_1,p_2) + i\psi_i(x_1,p_2), \quad V(x) = V_r(x_1,p_2) + iV_i(x_1,p_2),$$
$$E = E_r + iE_i \tag{11}$$

and use them in (9) along with the transformation (5). After separating the real and imaginary parts in the resultant expression and equating them to zero separately, one obtains [9] a pair of coupled equations in ψ_r and ψ_i. After using the ansatz,

$$\psi(x) = \psi_r + \mathrm{i}\psi_i = \exp[g(x)], \tag{12}$$

with $g(x) = g_r(x_1,p_2) + ig_i(x_1,p_2)$ for the solution ψ, these equations for g_r and g_i become [9]

$$2g_{r,x_1x_1} - 2g_{i,x_1}^2 + 2g_{r,x_1}^2 + (E_r - V_r) = 0 \tag{13a}$$

$$2g_{i,x_1x_1} + 4g_{i,x_1}g_{r,x_1} + (E_i - V_i) = 0. \tag{13b}$$

Using eqs (13a) and (13b), eigenvalues and eigenfunctions are obtained for a variety of complex potentials [10,14]. Some general remarks on these results are in order:

(1) In the present approach the eigenvalues remain real as long as the potential parameters are considered as real.

(2) The results obtained for the complex quartic and sextic potentials in the present approach are quite general and they are reduced, as special cases, to the ones obtained [15] for their respective $\mathcal{PT}$-symmetric versions. However, our results are found to contain a lesser number of constraining relations for the existence of real eigenvalue spectra. As a matter of fact, in the present approach there are more than one ways to achieve the real eigenvalues for complex potentials. In this respect the present approach is rich enough to describe the physical reality.

(3) In some cases, beside the ground state, excited states are also studied in the present framework not only in one [9,14] but also in higher dimensions [16]. Further, several interesting conclusions are derived when the present approach is applied to study the supersymmetric quantum mechanics [11].

3. Chevreul (hand-held) pendulum: Role of consciousness in the classical domain

In 1833, the French chemist Michel-Eugene Chevreul [17] wrote a remarkable paper on the experiments and interpretation of the 'magical' or hand-held pendulum (henceforth termed as 'Chevreul pendulum'). In fact, when a pendulum consisting of a heavy body (say an iron ring) attached to one of the ends of a flexible thread and of which the other end is held by an 'unmoving' hand is allowed to oscillate above certain substance (say water, a piece of metal or a living Being), then the presence of the latter induces the pendulum's oscillation in spite of the fact that the arm remains immobile. What has been found are the unusual properties of the inanimate object like the ones that the pendulum swings back and forth or in a circular path depending upon what the holder is thinking, i.e. the nature of thought reflects in the mode of oscillation. Chevreul, in fact, arrived at several interesting conclusions (and the same were communicated to Ampere) for which physics does not seem to have any explanation whatsoever even today. However, a detailed study of these data carried out by psychologists Ansfield and Wegner [18] ruled out any kind of mystical explanation. The first complete English translation of this important document along with an open letter written to A M Ampere (the top ranking physicist of that time) has recently been published by Spitz and Marcuard [19] in *Skeptical Inquirer*. Chevreul also highlighted several other phenomena of this category in his open letter.

With a view to understand such consciousness-manifesting phenomena, it appears that a parameter responsible for the hidden thought process or for the self-organizing mechanism needs to be incorporated in the theory [20,21]. As a modest effort in this direction, in the next section, we not only attribute the physical meanings of the fictitious (imaginary) variables x_2 and p_2 in the extended complex configuration space to this aspect of the data, but also consider their dependence on a variable μ (accounting for the growth of the thought process), introduced for the first time in the theory and treated on par with the time variable t.

4. Hamilton's principle of least (conscious) action in the extended complex configuration space

In order to formulate Hamilton's principle of least action when certain subjective component (attributed to some nonphysical aspects in the phenomenon like consciousness) in the data is involved, we introduce the dependence of the coordinate q and the velocity $p\ (=\dot{q})$ on another variable μ treated at par with and in addition to the time variable t in the transformation (4) as

$$\begin{aligned} q(t,\mu) &= q_1(t) + iq_2(\mu), \\ p(t,\mu) &= p_1(t) + ip_2(\mu), \end{aligned} \tag{14}$$

where $p_1 = \dot{q}_1$ and $p_2 = (\partial q_2/\partial\mu) = \bar{q}_2$ [22]. For simplicity we have not considered the dependence of q_2 and p_2 on t which is totally taken care of by the physical variables (q_1, p_1) and note that $(\partial q_1/\partial \mu) = (\partial p_1/\partial \mu) = 0$. Further, we write the complex Lagrangian of the system as

$$L(q, p, t, \mu) = L_1(q_1, p_1, q_2, p_2, t, \mu) + IL_2(q_1, p_1, q_2, p_2, t, \mu). \tag{15}$$

In other words, the pair (q_1, p_1) corresponds to the physical reality contained in the data and the pair (q_2, p_2) will account for the nonphysical reality if involved in the data. However, in the complex action K defined as

$$K = \int_{t_1}^{t_2} \int_{\mu_1}^{\mu_2} L(q_1(t) + iq_2(\mu), p_1(t) + ip_2(\mu), t, \mu)\mathrm{d}\mu \cdot \mathrm{d}t, \tag{16}$$

an interplay between the two realities in Nature will appear. Moreover, for a complete understanding of the phenomenon an account of both the realities is necessary.

In (15) we have split the complex Lagrangian L into real and imaginary parts where I instead of i is used with the imaginary part L_2. This is mainly because there is no reason as to why the dreaming of an object be the same as the dreaming of an action on it. They might belong to different imaginary planes. However, both i and I conform to $I^2 = i^2 = -1$. In fact, Carroll [23] has recently sought several interesting implications of such a choice of different imaginaries in his study of complex signals. In mathematical terms, we are considering different imaginaries with a complex number system and with the mappings on it. As we shall see, this is done at the cost of analyticity of the complex Lagrangian function L.

Before proceeding further, we remark here on the situation when $I \neq i$ [24]. Note that for the case when $q_1 = 0$, $q_2 = 1$, $L_1 = \mathbb{Z}$ (zero transformation) and $L_2 = \mathbb{1}$ (identity transformation), I will map i alone to give the image $I(i)$, i.e., I as some function of i. Now, if $I(i)$ is an even function of i, then this will result into the function I to be a real one and so is the transformation $L = L_1 + IL_2$. Therefore, for a complex transformation, $I(i)$ has to be an odd function of i and for this case $I(i)$ in its most general form, can be written as $I(i) = ai + b$ with a and b as real constants. However, the property $I(-i) = -I(i)$ implies $b = 0$, leaving behind $I(i) = ai$. Thus, without any loss of generality the constant a can be chosen as unity and perhaps for this reason $I = i$ is used throughout the theory of functions of complex variable. In the present case, however, neither q_1 can be zero nor can the real part L_1 be a zero transformation since we are basically building up a theory on the ground of physical reality. Therefore, $I \neq i$ is considered in the present approach.

First note from (14) that

$$(\partial/\partial q) = (\partial/\partial q_1) - i(\partial/\partial q_2), \quad (\partial/\partial p) = (\partial/\partial p_1) - i(\partial/\partial p_2), \tag{17}$$

and then, following the standard procedure based on calculus of variation [25], we proceed to find the stationary value of the complex functional (16). For this purpose, we introduce arbitrary parameters α with reference to t-variable and β with reference to μ-variable and write the integral (16) as

$$K(\alpha, \beta) = \int_{t_1}^{t_2} \int_{\mu_1}^{\mu_2} L(q_1(t, \alpha) + iq_2(\mu, \beta), p_1(t, \alpha) + ip_2(\mu, \beta), t, \mu)\mathrm{d}\mu \cdot \mathrm{d}t, \tag{18}$$

and demand

$$\left(\frac{\partial K}{\partial \alpha}\right)_{\substack{\alpha=0\\ \beta=0}} = 0, \quad \left(\frac{\partial K}{\partial \beta}\right)_{\substack{\alpha=0\\ \beta=0}} = 0$$

for the stationary action K. These conditions lead to [24] the following evolution equations with respect to the complex variables q and p:

$$\frac{\partial L}{\partial q} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial p}\right) = 0, \quad \frac{\partial L}{\partial q} - \frac{\partial}{\partial \mu}\left(\frac{\partial L}{\partial p}\right) = 0. \tag{19}$$

Now, using (15) and (17) and for $I \neq i$ case, eq. (19) yields the following set of eight equations:

$$\frac{\partial L_1}{\partial q_1} - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial p_1}\right) = 0; \quad \frac{\partial L_1}{\partial q_2} - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial p_2}\right) = 0, \tag{20a, b}$$

$$\frac{\partial L_2}{\partial q_1} - \frac{\partial}{\partial t}\left(\frac{\partial L_2}{\partial p_1}\right) = 0; \quad \frac{\partial L_2}{\partial q_2} - \frac{\partial}{\partial t}\left(\frac{\partial L_2}{\partial p_2}\right) = 0, \tag{20c, d}$$

$$\frac{\partial L_1}{\partial q_1} - \frac{\partial}{\partial \mu}\left(\frac{\partial L_1}{\partial p_1}\right) = 0; \quad \frac{\partial L_1}{\partial q_2} - \frac{\partial}{\partial \mu}\left(\frac{\partial L_1}{\partial p_2}\right) = 0, \tag{21a, b}$$

$$\frac{\partial L_2}{\partial q_1} - \frac{\partial}{\partial \mu}\left(\frac{\partial L_2}{\partial p_1}\right) = 0; \quad \frac{\partial L_2}{\partial q_2} - \frac{\partial}{\partial \mu}\left(\frac{\partial L_2}{\partial p_2}\right) = 0. \tag{21c, d}$$

Similarly, for $I = i$, eq. (19) provides the following set of four equations:

$$\frac{\partial L_1}{\partial q_1} + \frac{\partial L_2}{\partial q_2} - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial p_1}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_2}{\partial p_2}\right) = 0, \tag{22a}$$

$$\frac{\partial L_2}{\partial q_1} - \frac{\partial L_1}{\partial q_2} - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial p_1}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial p_2}\right) = 0, \tag{22b}$$

$$\frac{\partial L_1}{\partial q_1} + \frac{\partial L_2}{\partial q_2} - \frac{\partial}{\partial \mu}\left(\frac{\partial L_1}{\partial p_1}\right) - \frac{\partial}{\partial \mu}\left(\frac{\partial L_2}{\partial p_2}\right) = 0, \tag{23a}$$

$$\frac{\partial L_2}{\partial q_1} - \frac{\partial L_1}{\partial q_2} - \frac{\partial}{\partial \mu}\left(\frac{\partial L_2}{\partial p_1}\right) + \frac{\partial}{\partial \mu}\left(\frac{\partial L_1}{\partial p_2}\right) = 0, \tag{23b}$$

which, after using the Cauchy–Riemann conditions for a function of two complex variables, namely

$$\frac{\partial L_1}{\partial q_1} = \frac{\partial L_2}{\partial q_2}, \quad \frac{\partial L_1}{\partial q_2} = -\frac{\partial L_2}{\partial q_1};$$
$$\frac{\partial L_1}{\partial p_1} = \frac{\partial L_2}{\partial p_2}, \quad \frac{\partial L_1}{\partial p_2} = -\frac{\partial L_2}{\partial p_1}, \tag{24}$$

reduce to these simpler versions as

$$\frac{\partial L_1}{\partial q_1} - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial p_1}\right) = 0; \quad \frac{\partial L_2}{\partial q_1} - \frac{\partial}{\partial t}\left(\frac{\partial L_2}{\partial p_1}\right) = 0, \tag{25a, b}$$

$$\frac{\partial L_1}{\partial q_1} - \frac{\partial}{\partial \mu}\left(\frac{\partial L_1}{\partial p_1}\right) = 0; \quad \frac{\partial L_2}{\partial q_1} - \frac{\partial}{\partial \mu}\left(\frac{\partial L_2}{\partial p_1}\right) = 0. \tag{26a, b}$$

Note that for a function of two complex variables the use of other by-product equations [26] (in addition to (24)) will further reduce the four equations in (25) and (26) to two equations, one corresponds to each (25) and (26).

With regard to the above equations derived from the complex action integral (16), the following remarks are in order:

(i) The system of equations obtained for $I \neq i$ (see eqs (20) and (21)) is richer than the one obtained for $I = i$ (see eqs (25) and (26)) in the sense that equations in the latter appear as certain combinations of the equations in the former. Further, in the former case the effect of consciousness is fine-tuned. It is however done at the cost of the analyticity property of the underlying complex function L and the same in fact can be relaxed when the subjective component in the data is present. If the subjectivity in the data is attributed to the element of consciousness (accounted here by the variable μ), then naturally the consciousness of a living Being cannot be fragmented as is the case with the time variable. Therefore, analyticity property of a function is bound to break down, say for an open system or in the presence of subjectivity in the data.

(ii) It is interesting to note that eq. (20a) involves only the real parts of q, p and L and that eq. (21d) involves only the imaginary parts of q, p and L. This means that these equations account respectively for the evolutions of physical and nonphysical aspects exclusively, whereas other equations in (20) and (21) do describe the evolution of a system with mixed situations (as is the case with Chevreul pendulum) in which the space, time and consciousness play their roles in certain proportions.

(iii) Note that eqs (20b)–(20d) and (21a)–(21c) in some sense are the space-time-consciousness evolution equations. One can ask as to how the consciousness evolves in a system like Chevreul pendulum. (In fact Chevreul pendulum is one example; otherwise in the day-to-day life there arise many situations where the evolution of the system takes place [20,21,27] with respect to the space, time and consciousness simultaneously and the evolution with respect to the latter in general might affect the outcome of action. However, we avoid going into such details here.) The Chevreul pendulum is the case of a conscious support. One can as well have a pendulum with a conscious bob, say a live bird is suspended from a rigid (inanimate) support and performing oscillations [20]. With regard to the evolution of a system with respect to μ, note that an object is not instantly perceived by the brain of a perceiver even after it is seen by his eyes; it takes certain fraction of a second for this purpose. On the other hand, modern brain-studies have revealed that the thought process in conjunction with the memory to realize the geometry and other finer details of an external object, do not come into action instantaneously via brain-cells and neural networks, rather it also takes certain fraction of a second. Moreover, for an unconscious state of mind of a person such a mechanism is not possible and his hand can be as good as a rigid (inanimate) support in the Chevreul pendulum. Thus, it is reasonable to attribute such a time-loss in the highly complex perception process (which in turn translates in bringing the faculties of thinking and memory [27] into action) to the consciousness evolution described by the variable μ in eqs (20) and (21).

5. Summary and conclusions

After highlighting various methods of complexification of a real Hamiltonian, the viability of the one, i.e., Method C4 (based on the complexification of physical variables x and p) is emphasized throughout this paper. For this purpose, the concepts of extended complex configuration and phase spaces are explored to study certain aspects of classical and quantum mechanics of complex systems. It is observed [9–11] that the $\mathcal{PT}$ symmetry and the results derived thereof appear as special cases in the present framework. It is argued that there is enough scope in the present approach to find a clue about the openness of a physical system [28] or about the role of nonphysical reality in the data collected using Chevreul pendulum. To this effect, a generalization of Hamilton's principle of least action in the extended complex configuration space is discussed. The Euler–Lagrange equations so derived may throw some light in explaining the data obtained using Chevreul pendulum. Such studies are in progress.

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References

[1] For a Review on $\mathcal{PT}$-symmetric Hamiltonians see, for example, C M Bender, *Rep. Prog. Phys.* **70**, 947 (2007) and the references therein
C M Bender, *Contemp. Phys.* **46**, 277 (2005) and the references therein

[2] H Feshbach, C E Porter and V F Weisskopf, *Phys. Rev.* **96**, 488 (1954)
[3] See, for example, R K Colegrave, P Croxson and M A Mannan, *Phys. Lett.* **A131**, 407 (1988)
[4] N N Rao, B Buti and S B Khadkikar, *Pramana – J. Phys.* **27**, 497 (1986)
B Buti, N N Rao and S B Khadkikar, *Phys. Scr.* **34**, 729 (1986)
[5] C-D Yang, *Chaos, Solitons and Fractals* **33**, 1073 (2007)
[6] A L Xavier and M A M Aguiar, *Ann. Phys.* (*N.Y.*) **252**, 45 (1996)
[7] R S Kaushal and H J Korsch, *Phys. Lett.* **A276**, 47 (2000)
[8] R S Kaushal and Shweta Singh, *Ann. Phys.* (*N.Y.*) **288**, 253 (2001)
Shweta Singh and R S Kaushal, *Phys. Scr.* **67**, 181 (2003)
[9] R S Kaushal, *J. Phys. A: Math. Gen.* **34**, L709 (2001)
[10] R S Kaushal and Parthasarathi, *J. Phys. A: Math. Gen.* **35**, 8743 (2002)
Parthasarathi and R S Kaushal, *Phys. Scr.* **68**, 115 (2003)
[11] Parthasarathi, D Parashar and R S Kaushal, *J. Phys. A: Math. Gen.* **37**, 781 (2004)
[12] C M Bender and S Boettcher, *Phys. Rev. Lett.* **80**, 5243 (1998)
C M Bender, S Boettcher and P N Meissinger, *J. Math. Phys.* **40**, 2201 (1999)
[13] A Mustafazadeh, *J. Math. Phys.* **43**, 205; 2814 (2002)
[14] Parthasarathi, *A study of complex phase space approach to quantum mechanics and supersymmetric quantum mechanics*, Ph.D. Thesis (University of Delhi, 2004) (Unpublished)
[15] F Cannata, M Ioffe, R Roy Choudhury and P Roy, *Phys. Lett.* **A281**, 305 (2001)
[16] Ram Mehar Singh, Fakir Chand and S C Mishra, *Pramana – J. Phys.* **72**, 647 (2009)
[17] E Chevreul, Lettre' A M Ampere', 'sur ume classe particuliere de mouvemens muscularies', *Revue des Deus Mondes* **2**, 249 (1833)
[18] M E Ansfield and D M Wegner, The feeling of doing, in: *The psychology of action: Linking cognition and motivation to behaviour* edited by P M Gollwitzer and J A Bargh (Guilford Publication Press, N.Y., 1996) Ch. 21
[19] H H Spitz and Y Marcuard, Chevreul's report on the mysterious oscillations of the hand-held pendulum: A French Chemist's 1833 Open Letter to Ampere (Translation from French), *Skeptical Inquirer* **25**, 35 (2001), No. 4. This is the magazine for Science and Reason published by the International Organization, 'Committee for Scientific Investigations of Claim of the Paranormal' (CSICOP)
[20] R S Kaushal, *Indian Phil. Quarterly (Poona)* **29**, 65 (2002)
[21] A Raymond Penner, *Rep. Prog. Phys.* **66**, 131 (2003)
[22] Note that the realization of an object by a conscious person is not instantaneous. It rather takes a fraction of second before the person finally realizes the object and during this time not only the physiological processes continue inside the body but his consciousness also evolves to become appropriately intense.
[23] J E Carroll, *Euro. J. Phys.* **28**, 1151 (2007)
[24] Radhey Shyam Kaushal, *Ganita Bharati* **30(1)**, 99 (2008)
[25] See, for example, H Goldstein, *Classical mechanics* (Addison-Wesley Publishing Co., 1981) Ch. 2
[26] V S Vladimirov, *Methods of the theory of functions of many complex variables* (MIT Press, Cambridge, MA, 1966)
[27] Radhey Shyam Kaushal, The science of philosophy, *Theory of fundamental processes in humamn behaviour and experiences* (to be published by D.K. Print World (P) Ltd., New Delhi)
[28] See, for example, R Ramanathan, Quantum mechanics of open systems as a stochastic process, in: *Stochastic processes and their applications* edited by A Vijay Kumar and M Sreenivasan (Narosa Pub. House, New Delhi, 1999) p. 317