

Raychaudhuri equation in quantum gravitational optics

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Abstract. The equation of Raychaudhuri is one of the key concepts in the formulation of the singularity theorems introduced by Penrose and Hawking. In the present article, taking into account QED vacuum polarization, we study the propagation of a bundle of rays in a background gravitational field through the perturbative deformation of Raychaudhuri's equation. In a sense, this could be seen as another semiclassical study in which geometry is treated classically but matter (which means the photon here) is allowed to exhibit quantum characteristics that are encoded in its coupling to the background curvature.

Keywords. Raychaudhuri equation; quantum gravitational optics; vacuum polarization.

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1. Introduction

Vacuum polarization is a quantum field-theoretic effect in which the photon exists as a virtual e^+e^- pair for a short time. This virtual transition assigns photons with an effective size of $\mathcal{O}(\lambda_c)$, where λ_c is the Compton wavelength of the electron [1]. In a curved background, the photon propagation will obviously be affected by the gravitational field if the typical scale of the space-time curvature L is comparable to λ_c . A simple analysis of null rays in classical general relativity implies that a curved space-time, compared to the flat case, could be treated as an optical medium with a refractive index [2]. The further addition of QED effects, in the context of semi-classical gravity, leads to interesting phenomena such as dispersive effects and the polarization-dependent propagation of photons. One of the main consequences is the modification of the light cone in such a way that QED photons do not propagate along null geodesics of the background geometry. They propagate instead along null geodesics of an effective geometry. Quantum characteristics of photon propagation in a curved background open up a whole new field. Coined quantum gravitational optics (QGO) [3], it started with the pioneering paper by Drummond and Hathrell and continued with detailed studies of the same effect for

specific space-times [4–6]. The modification of the underlying space-time geometry, felt by photons due to QED interactions, has led to some unexpected results.

Here we shall study the geometric properties of *physical* null congruences which are families of physical null rays determined by the effective metric $\mathcal{G}_{\mu\nu}$. These are then distinct from the properties of the *geometric* null congruences characterized by the space-time metric $g_{\mu\nu}$. Such congruences are special in the sense that the modification of the light cone in a local inertial frame (LIF) can be described as $(\eta_{(a)(b)} + \alpha^2 \sigma_{(a)(b)}(R)) k^{(a)} k^{(b)} = 0$, where $\eta_{(a)(b)}$ is the Minkowski metric, α is the fine structure constant and $\sigma_{(a)(b)}(R)$ depends on the Riemann curvature at the origin of the LIF. In a one-loop approximation, these photons travel with unit velocity. For on-shell photons, some of the Newman–Penrose (NP) scalars play the role of optical scalars. Their propagation along the rays is governed by the known Raychaudhuri equation. We show that these scalars must be effectively modified to describe *physical* null congruences and find that a new set of NP scalars contribute to the definition of these *effective optical scalars*. Next, we discuss the modified Raychaudhuri equation which governs the propagation of these scalars.

2. Vacuum polarized photon propagation in a curved background

The effect of one-loop vacuum polarization on the photon propagation in a fixed curved background space-time is represented by the following effective action derived by Drummond and Hathrell [1]:

$$\Gamma = \int dx \sqrt{-g} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{m^2} \left(a R F_{\mu\nu} F^{\mu\nu} + b R_{\mu\nu} F^{\mu\lambda} F^{\nu\lambda} + c R_{\mu\lambda\nu\rho} F^{\mu\lambda} F^{\nu\rho} \right) \right]. \quad (1)$$

Here, $a = -\frac{1}{144} \frac{\alpha}{\pi}$, $b = \frac{13}{360} \frac{\alpha}{\pi}$ and $c = -\frac{1}{360} \frac{\alpha}{\pi}$ where α is the fine structure constant and m is the electron mass. The notable feature in the above action is the direct coupling of the electromagnetic field to the curvature tensor which in effect violates the strong equivalence principle. It should also be noted that since the photons are treated as test particles the backreaction has been assumed to be negligible. The effective equation of motion can be obtained by varying Γ with respect to A_ν as follows:

$$D_\mu F^{\mu\nu} + \frac{1}{m^2} \left(2b R^\mu{}_\lambda D_\mu F^{\lambda\nu} + 4c R^{\mu\nu}{}_{\lambda\rho} D_\mu F^{\lambda\rho} \right) = 0. \quad (2)$$

There are some approximations under which this equation of motion was obtained. The first one is the low frequency approximation in the sense that the derivation is only applicable to wavelengths $\lambda > \lambda_c$. By this approximation we ignore terms in the effective action involving higher-order field derivatives. The second is a weak-field approximation for gravity. This implicitly means that the wavelengths $\lambda_c \ll L$ are considered.

Employing the geometric optics approximation, the electromagnetic field can be written in the form $A_\mu = A a_\mu e^{i\theta}$, where A is the amplitude and a_μ is the polarization vector. If we now identify the wave vector as $k_\mu = \partial_\mu \theta$, the equation governing

the components of this vector field for a photon with space-like, normalized polarization vector i.e, $a_\mu a^\mu = -1$, can be read from (2) as

$$k^2 + \frac{2b}{m^2} R_{\mu\lambda} k^\mu k^\lambda - \frac{8c}{m^2} R_{\mu\nu\lambda\rho} k^\mu k^\lambda a^\nu a^\rho = 0. \quad (3)$$

Equation (3) is an effective light cone equation which re-expressed in terms of the Weyl tensor is given by

$$k^2 + \frac{(2b + 4c)}{m^2} R_{\mu\lambda} k^\mu k^\lambda - \frac{8c}{m^2} C_{\mu\nu\lambda\rho} k^\mu k^\lambda a^\nu a^\rho = 0. \quad (4)$$

The corresponding momentum of the photon, i.e. p^μ , is the tangent vector to the light ray x^μ i.e, $p^\mu = \frac{d}{ds} x^\mu(s)$. Equation (4), being quadratic and homogenous, could be written in the following form:

$$\mathcal{G}^{\mu\nu} k_\mu k_\nu = 0. \quad (5)$$

This shows that, at this level of approximation, QGO can be characterized as a bimetric theory. The physical light cones are determined by the effective metric $\mathcal{G}_{\mu\nu}$, and are distinct from the geometrical null cones which are fixed by the space-time metric $g_{\mu\nu}$ (Note that indices are always raised and lowered using the space-time metric $g_{\mu\nu}$). Some implications of this bimetricity, including gravitational birefringence and superluminal speed of light, have been discussed in [7]. In the present study, we focus on modifications produced by vacuum polarization effects on the propagation of light rays in a curved background.

3. Abreast rays, optical scalars and Raychaudhuri equation

The physical congruence is specified by the integral curves γ of the vector field k^μ parametrized by the parameter u and scaled such that $\nabla_k u = 1$. The connecting vector field q^μ is introduced [8] in order to examine the relation between the neighbouring curves of the congruence. Defined along a particular curve γ , it characterizes the displacement from a point $P \in \gamma$ to a point P' on a neighbouring curve, where P and P' have the same parameter value u . Mathematically, this means that q^μ is being Lie transported along the curve by the vector field k^μ , i.e.

$$\mathcal{L}_k q^\mu = 0, \quad \text{or} \quad \nabla_k q^\mu = \nabla_q k^\mu. \quad (6)$$

Here the notation $\nabla_X = X^\alpha \nabla_\alpha$ for directional covariant derivative is used. Using the NP tetrad formalism and following closely the notations of [8] and [9], as a first step, we choose a null tetrad with l^μ to be the null vector along the photon momentum. Let a^μ and b^μ be space-like transverse vectors (later they will be identified with the polarization vectors). Define the complex null vectors m^μ and \bar{m}^μ by $m^\mu = \frac{1}{\sqrt{2}}(a^\mu + ib^\mu)$ and $\bar{m}^\mu = \frac{1}{\sqrt{2}}(a^\mu - ib^\mu)$ and add to this set another real null vector n^μ . All the null vectors satisfy the usual NP orthogonality conditions

$$l \cdot m = l \cdot \bar{m} = n \cdot m = n \cdot \bar{m} = 0, \quad (7)$$

and

$$l \cdot l = n \cdot n = m \cdot m = \bar{m} \cdot \bar{m} = 0. \quad (8)$$

The normalization conditions

$$l \cdot n = -m \cdot \bar{m} = 1 \quad (9)$$

are also imposed. We assign a vierbein $e_{(a)}^\mu$ to this tetrad [10] as

$$\begin{aligned} e_{(1)} = e^{(2)} = l, \quad e_{(2)} = e^{(1)} = n, \\ e_{(3)} = -e^{(4)} = m, \quad e_{(4)} = -e^{(3)} = \bar{m}, \end{aligned} \quad (10)$$

with the frame metric of the form

$$\eta_{(a)(b)} = e_{(a)}^\mu e_{\mu(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (11)$$

In the tetrad formalism, various quantities called ‘Ricci rotation coefficients’, $\gamma_{(c)(a)(b)}$, are employed to account for the tetrad covariant differentiation. These quantities can be defined as

$$e_{(a)\mu;\nu} = e_{\mu}^{(c)} \gamma_{(c)(a)(b)} e_{\nu}^{(b)}. \quad (12)$$

In the NP formalism, they are called ‘spin coefficients’ and are designated by the following special symbols [9]:

$$\begin{aligned} \kappa &= \gamma_{(3)(1)(1)}; & \rho &= \gamma_{(3)(1)(4)}; & \epsilon &= \frac{1}{2} (\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}); \\ \sigma &= \gamma_{(3)(1)(3)}; & \mu &= \gamma_{(2)(4)(3)}; & \gamma &= \frac{1}{2} (\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)}); \\ \lambda &= \gamma_{(2)(4)(4)}; & \tau &= \gamma_{(3)(1)(2)}; & \alpha &= \frac{1}{2} (\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)}); \\ \nu &= \gamma_{(2)(4)(2)}; & \pi &= \gamma_{(2)(4)(1)}; & \beta &= \frac{1}{2} (\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)}). \end{aligned} \quad (13)$$

A general rule for obtaining the complex conjugate of any quantity is to replace the index (3), wherever it occurs, by the index (4) and vice versa.

The NP formalism is most advantageous when the null tetrads introduced can be completely tied to the geometry of the problem. Otherwise, some freedom remains in the choice of the basis as a result of subjecting it to the Lorentz transformation. This has the effect that many of the quantities involved in the calculation are of the same nature as gauge quantities whose values transform in certain ways as the basis frame is varied in accordance with the remaining freedom. In QGO, one null direction is singled out and that is the unperturbed photon momentum which is fixed in the direction of the null vector l^μ . Therefore the allowed transformations would be those which leave the l^μ direction unchanged and preserve the underlying orthogonality and normalization conditions. These are classified as follows:

$$\begin{aligned} \text{class I: } & l \mapsto l; \quad m \mapsto m + al; \quad \bar{m} \mapsto \bar{m} + a^*l; \quad n \mapsto n + a^*m + a\bar{m} + aa^*l; \\ \text{class II: } & l \mapsto A^{-1}l; \quad n \mapsto An; \quad m \mapsto e^{i\theta}m; \quad \bar{m} \mapsto e^{-i\theta}\bar{m}, \end{aligned} \quad (14)$$

where a is a complex function and A and θ are two real functions on the manifold. We now proceed to write down eq. (6) in the above tetrad basis. Before doing so, let us employ all the gauge freedom that we have to fix the tetrad basis. As the tetrad frame is transformed, different tetrad components will be subject to changes. Starting with the l vector as the velocity vector along the geometric congruence, we have

$$\nabla_l l^\mu = (\epsilon + \epsilon^*) l^\mu - \kappa \bar{m}^\mu - \kappa^* m^\mu \propto l^\mu \implies \kappa = 0. \quad (15)$$

Moreover, if they are affinely parametrized,

$$\nabla_l l^\mu = 0 \implies \kappa = \epsilon = 0. \quad (16)$$

If $\kappa = 0$, the latter requirement can be met by a class II rotation which will not affect the direction of l nor of the initial vanishing of κ . This is because we have

$$\begin{aligned} \gamma_{(c)(a)(b)} &= e_{(c)}^k e_{(a)k;i} e_{(b)}^i, \\ \kappa = \gamma_{(3)(1)(1)} &= m^k l_{k;i} l^i \mapsto e^{i\theta} m^k (A^{-1}l)^i (A^{-1}l_{k;i}) \\ &= A^{-2} e^{i\theta} \kappa + e^{i\theta} (m \cdot l) A^{-1} (\nabla_l A^{-1}) = A^{-2} e^{i\theta} \kappa, \end{aligned}$$

$$\begin{aligned} \epsilon &= \frac{1}{2} [\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}] = \frac{1}{2} [n^k l_{k;i} l^i + m^k \bar{m}_{k;i} l^i] \mapsto \\ &= \frac{1}{2} [An^k (A^{-1}l_{k;i}) A^{-1}l^i + e^{i\theta} m^k (e^{-i\theta} \bar{m})_{k;i} A^{-1}l^i] \\ &= \frac{1}{2} [n^k l_{k;i} l^i A^{-1} + (n \cdot l) A_{;i}^{-1} l^i + m^k \bar{m}_{k;i} l^i A^{-1} - i\theta_{;i} (m \cdot \bar{m}) A^{-1} l^i] \\ &= \frac{1}{2} A^{-1} \epsilon - \frac{1}{2} A^{-2} \nabla_l A + \frac{1}{2} i A^{-1} \nabla_l \theta. \end{aligned} \quad (17)$$

By a suitable rotation of class I, it could be arranged so that $\pi = \gamma_{(2)(4)(1)}$ vanishes. Thus

$$\begin{aligned} \pi = \gamma_{(2)(4)(1)} &= n^k \bar{m}_{k;i} l^i \mapsto (n^k + a^* m^k + a \bar{m}^k + aa^* l^k) (\bar{m} + a^* l)_{k;i} l^i \\ &= \pi + 2a^* \epsilon + (a^*)^2 \kappa + \nabla_l a^*. \end{aligned} \quad (18)$$

In a similar fashion, we find that the spin coefficients κ and ϵ transform as $\kappa \mapsto \kappa$ and $\epsilon \mapsto \epsilon + a^* \kappa$ in this class. So the gauge, that we are working in, is the one in which $\kappa = \epsilon = \pi = 0$. After such a rotation, the newly oriented vectors n, m and \bar{m} will remain unchanged as they undergo parallel transport along l . This could be invoked through the following relations:

$$\begin{aligned} \nabla_l m^\nu &= \pi^* l^\nu - \kappa n^\nu + (\epsilon - \epsilon^*) m^\nu, \\ \nabla_l n^\nu &= \pi^* \bar{m}^\nu + \pi m^\nu - (\epsilon + \epsilon^*) n^\nu, \end{aligned} \quad (19)$$

with the right-hand sides vanishing in the above gauge. Now we have exploited all the gauge freedom that we had and are ready to write eq. (6) in the above gauge. We only consider states which propagate with a well-defined polarization. For two transverse polarizations expressed in terms of the vectors m^μ and \bar{m}^μ , representing the left and the right handed circular polarizations, we have

$$\begin{aligned} \nabla_k q^\nu = \nabla_q \left[l^\nu - \frac{1}{m^2} (b + 2c) R^{\lambda\nu} l_\lambda \right. \\ \left. \mp \frac{2c}{m^2} C^\nu_{\lambda\sigma\kappa} l^\sigma (m^\lambda \pm \bar{m}^\lambda) (m^\kappa \pm \bar{m}^\kappa) \right]. \end{aligned} \quad (20)$$

On multiplying by $e_\nu^{(c)}$, we get

$$\nabla_k q^{(c)} = \left(\eta^{(c)(a)} + \frac{1}{m^2} A^{(c)(a)} \right) q^{(b)} \gamma_{(a)(1)(b)} + q_{(a)} \gamma^{(a)(c)(b)} k_{(b)}, \quad (21)$$

where

$$\begin{aligned} A^{(c)(a)} &:= -(b + 2c) R^{(c)(a)} \mp (2c) M^{(c)(a)}, \\ M^{(c)(a)} &:= C^{(c)(3)(a)(3)} + C^{(c)(4)(a)(4)} \pm (C^{(c)(3)(a)(4)} + C^{(c)(4)(a)(3)}). \end{aligned} \quad (22)$$

The specifically *gravitational* birefringence shows up here in $M^{(a)(c)}$ (we note that $A^{(c)(a)} = A^{(a)(c)}$). Equation (21) is a system of coupled differential equations describing the propagation of the tetrad components of the connecting vector along the velocity vector k^μ . The second term in (21) appears as a consequence of the propagation of the null tetrad along the *physical* ray (with the tangent vector k^μ) i.e. $q^\nu \nabla_k e_\nu^{(c)}$. We introduce different tetrad components of q^μ in the following way:

$$q^\mu = gl^\mu + \xi \bar{m}^\mu + \bar{\xi} m^\mu + hn^\mu. \quad (23)$$

Thus

$$q^{(1)} = q_{(2)} = g; \quad q^{(2)} = q_{(1)} = h; \quad q^{(3)} = -q_{(4)} = \bar{\xi}; \quad q^{(4)} = -q_{(3)} = \xi. \quad (24)$$

Explicitly for $c = 4$, eq. (21) gives

$$\begin{aligned} \nabla_k q^{(4)} = \left(\eta^{(4)(3)} + \frac{1}{m^2} A^{(4)(3)} \right) \\ [q^{(2)} \gamma_{(3)(1)(2)} + q^{(3)} \gamma_{(3)(1)(3)} + q^{(4)} \gamma_{(3)(1)(4)}] \\ + \frac{1}{m^2} A^{(4)(4)} [q^{(2)} \gamma_{(4)(1)(2)} + q^{(3)} \gamma_{(4)(1)(3)} + q^{(4)} \gamma_{(4)(1)(4)}] \\ + \frac{1}{m^2} A^{(4)(2)} q^{(b)} \gamma_{(2)(1)(b)} - \frac{1}{m^2} A^{(2)(b)} q^{(a)} \gamma_{(a)(3)(b)} \end{aligned} \quad (25)$$

in which the last term is again the contribution due to null tetrad propagation along the physical ray discussed below eq. (22). In order to identify the geometrical and

physical content of this equation, a ‘covariant approach’ in which any complicated gauge behaviour is completely avoided must be chosen. For example, in eq. (25), the term $A^{(4)(4)}q^{(2)}\gamma_{(4)(1)(2)}$, having a complicated gauge behaviour, transforms into

$$[A^{(4)(4)} - 2aA^{(4)(2)} + a^2A^{(2)(2)}]q^{(2)}[\gamma_{(4)(1)(2)} + a^*\gamma_{(4)(1)(3)} + a\gamma_{(4)(1)(4)}] \quad (26)$$

under a class I transformation. To keep the formalism covariant, we remove the gauge-dependent terms by applying the extra condition that the neighbouring pair of rays obey the relation

$$q.k = 0. \quad (27)$$

We call such rays *abreast* [8]. This is a *primary* constraint and it means that

$$\begin{aligned} O(\alpha^0) : \quad q.l = 0 &\Rightarrow h = 0; \\ O(\alpha^1) : \quad A_{(1)(a \neq 2)} = 0 &\Rightarrow \Phi_{00} = \Phi_{01} = \Phi_{10} = \Psi_0 = \Psi_1 = 0. \end{aligned} \quad (28)$$

The parametrization independence of the abreast ray constraint can be seen by making the substitution $q^\mu \mapsto q^\mu + \Lambda k^\mu$. We also emphasize that, provided the abreast constraint (28) is satisfied, the tangent vector k^μ is given by

$$\begin{aligned} k_{(2)} &= 1 - \frac{1}{m^2}A_{(1)(2)}, \quad k_{(a \neq 2)} = 0, \\ k^2 &= O(\alpha^2), \quad \nabla_k k^\mu = O(\alpha^2). \end{aligned} \quad (29)$$

The abreast rays are *special* physical rays whose effective light cone condition, in a local inertial frame, is given by

$$(\eta_{(a)(b)} + \alpha^2\sigma_{(a)(b)}(R))k^{(a)}k^{(b)} = 0. \quad (30)$$

This should be compared with the general light cone modification (3) induced by the effective action (1) which is of order α .

3.1 Effective optical scalars

To study the evolution of the cross sections of (physical) bundles of rays, through their so-called *optical scalars* [8], we start with the covariant deformation of ξ :

$$\nabla_k \xi = -(\bar{\xi}\sigma + \xi\rho) + \frac{1}{m^2}A^{(4)(3)}(\bar{\xi}\sigma + \xi\rho) + \frac{1}{m^2}A^{(4)(4)}(\bar{\xi}\rho^* + \xi\sigma^*). \quad (31)$$

The transverse distances from a specified ray γ are (frame) observer-independent. Therefore, for abreast rays, optical scalars bear explicit physical interpretations. To simplify the case we consider propagation along k of the area of a small triangle formed by the points o, ξ_1 and ξ_2 (see figure 1). Thus we have

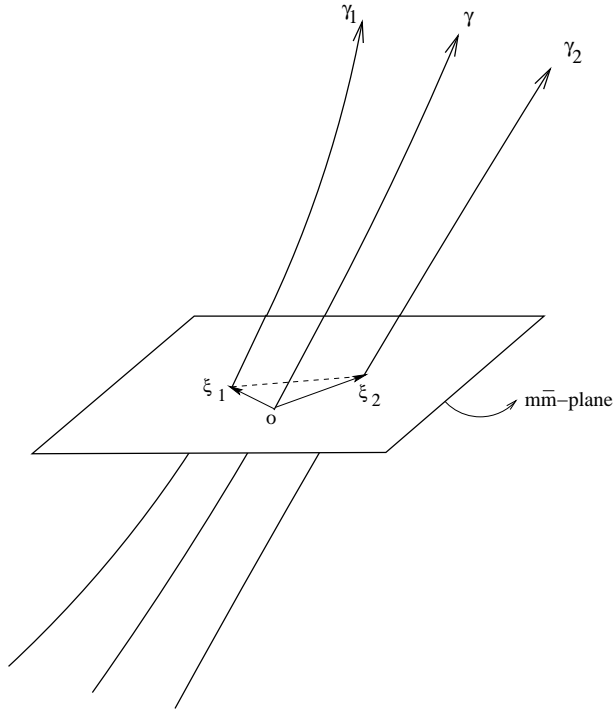


Figure 1. Three abreast rays of a congruence and the small triangle formed by two connecting vectors ξ_1 and ξ_2 , connecting the ray γ to the rays γ_1 and γ_2 in the $m\bar{m}$ -plane.

$$\begin{aligned} \nabla_k \delta A &= \nabla_k \left[\frac{i}{2} (\xi_1 \bar{\xi}_2 - \xi_2 \bar{\xi}_1) \right] \\ &= - \left[\rho - \frac{1}{m^2} (A^{(3)(4)} \rho + A^{(3)(3)} \sigma) + c.c \right] \delta A. \end{aligned} \quad (32)$$

The scalar quantity

$$-\text{Re} \left[\rho - \frac{1}{m^2} (A^{(3)(4)} \rho + A^{(3)(3)} \sigma) \right] \quad (33)$$

is called the expansion parameter θ_{eff} . Its role as a measure of the pattern convergence (or divergence) is clear from equation (32). This equation shows that in an effective theory, unlike in the classical case, the local effect of σ can also change the area. Another useful parameter, the ‘luminosity parameter’ L , is defined along a bundle of rays as $L^2 \propto \delta A$ and is related to θ_{eff} as follows (using equation (32)):

$$\nabla_k L = \theta_{\text{eff}} L. \quad (34)$$

In other words, the expansion parameter is the logarithmic derivative of the luminosity parameter. An alternative formula for θ_{eff} is given by $\frac{1}{2} k^\mu{}_{;\mu}$.

Setting θ_{eff} and the coefficient of $\bar{\xi}$ equal to zero in eq. (31), we obtain

$$\nabla_k \xi = -i \text{Im} \left[\rho - \frac{1}{m^2} \left(A^{(3)(4)} \rho + A^{(3)(3)} \sigma \right) \right] \xi. \quad (35)$$

Thus it can also be claimed that $\text{Im} \left[\rho - \frac{1}{m^2} \left(A^{(3)(4)} \rho + A^{(3)(3)} \sigma \right) \right]$ measures the twist in the bundle's cross-section and can be called the effective twist, ω_{eff} . The fact that this combination of scalars is a measure of the twist is consistent with the requirement that it measures the failure of k_μ to be hypersurface orthogonal. In the same spirit, if $\theta_{\text{eff}} = \omega_{\text{eff}} = 0$, the remaining part of eq. (31), i.e

$$\sigma + \frac{1}{m^2} \left(A^{(4)(3)} \sigma + A^{(4)(4)} \rho^* \right), \quad (36)$$

can be interpreted as the effective shear σ_{eff} . This is a measure of the distortion in the shape of the bundle's cross-section so that a circular cross-section transforms into an elliptic one.

3.2 Effective Raychaudhuri equation

In order to study the variation of the above effective optical scalars along the physical ray, one can examine the second derivative of the connecting vector q^μ , propagating along the wave vector k_μ , through the operation of ∇_k on eq. (1):

$$\begin{aligned} \nabla_k \nabla_k q^\mu &= \nabla_k (\nabla_q k^\mu) = \nabla_q \nabla_k k^\mu + R_{\lambda\nu\sigma}{}^\mu k^\lambda q^\nu k^\sigma \\ &= R_{\lambda\nu\sigma}{}^\mu k^\lambda q^\nu k^\sigma + O(\alpha^2). \end{aligned} \quad (37)$$

On multiplying by $e_\mu^{(c)}$ and taking the components corresponding to the tetrads subjected to parallel transport, we get

$$\nabla_k \nabla_k q^{(c)} = R_{(1)(3)(1)}{}^{(c)} \bar{\xi} + R_{(1)(4)(1)}{}^{(c)} \xi + O(\alpha^2). \quad (38)$$

We have neglected here the terms, including the product of curvature components, which would be suppressed by $O(\frac{R}{m^2})$. From eq. (31), we obtain the useful (2×2) matrix form as

$$\nabla_k \mathbf{Z} = \mathbf{P} \mathbf{Z}, \quad (39)$$

where

$$\mathbf{Z} = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \theta_{\text{eff}} - i\omega_{\text{eff}} & \sigma_{\text{eff}} \\ \sigma_{\text{eff}}^* & \theta_{\text{eff}}^* + i\omega_{\text{eff}}^* \end{pmatrix}. \quad (40)$$

For abreast rays, eq. (39) reduces to

$$\nabla_k \mathbf{P} = -\mathbf{P}^2. \quad (41)$$

Equation (41), written out in full, leads to the directional propagation of the effective optical scalars as

$$\nabla_k \sigma_{\text{eff}} = -2\theta_{\text{eff}} \sigma_{\text{eff}}, \tag{42}$$

$$\nabla_k \omega_{\text{eff}} = -2\theta_{\text{eff}} \omega_{\text{eff}}, \tag{43}$$

$$\nabla_k \theta_{\text{eff}} = \omega_{\text{eff}}^2 - \theta_{\text{eff}}^2 - |\sigma_{\text{eff}}|^2, \tag{44}$$

the last of which is the effective Raychaudhuri equation whose physically important consequences are discussed below.

4. Discussion

In the derivation of the classical version of the Raychadhuri equation [11,12], the variation of the tangent vector $k_{\mu;\nu}$ is projected [13] on the plane spanned by m^μ and \bar{m}^μ . Then the antisymmetric and symmetric parts of its trace are designated as expansion, vorticity and shear respectively. In the present modified version, we chose, through eq. (6), an equivalent method in which the variations of the connecting vector q^μ instead of the tangent vector have been considered and then the corresponding *effective* optical scalars have been defined. The invariants of the theory are easily seen in this way. For a luminosity parameter L , we see from eq. (43) that $\nabla_k (L^2 \omega_{\text{eff}}) = 0$. In other words, $q \cdot k$ and $L^2 \omega_{\text{eff}}$ are the constant geometric quantities along the congruence. Also, for two neighbouring rays of γ , whose connecting vectors q^μ and \tilde{q}^μ independently satisfy eq. (37), a symplectic invariant is assigned as follows:

$$q^\mu \nabla_k \tilde{q}_\mu - \tilde{q}^\mu \nabla_k q_\mu. \tag{45}$$

The constancy of expression (45) along γ is a consequence of the interchange symmetry of the curvature tensor and eq. (37):

$$\begin{aligned} \nabla_k (q^\mu \nabla_k \tilde{q}_\mu - \tilde{q}^\mu \nabla_k q_\mu) &= (\nabla_k q^\mu) (\nabla_k \tilde{q}_\mu) + q^\mu \nabla_k \nabla_k \tilde{q}_\mu \\ &\quad - (\nabla_k \tilde{q}^\mu) (\nabla_k q_\mu) - \tilde{q}^\mu \nabla_k \nabla_k q_\mu \\ &\approx q^\mu R_{\lambda\nu\sigma\mu} k^\lambda \tilde{q}^\nu k^\sigma - \tilde{q}^\mu R_{\lambda\nu\sigma\mu} k^\lambda q^\nu k^\sigma = 0. \end{aligned} \tag{46}$$

Considering only abreast rays, expression (45) can be written in the form

$$\begin{aligned} Z^\top (\nabla_k \tilde{Z}) - (\nabla_k Z^\top) \tilde{Z} &= Z^\top (P \tilde{Z}) - (P Z)^\top \tilde{Z} = Z^\top P \tilde{Z} - Z^\top P^\top \tilde{Z} \\ (\bar{\xi} \tilde{\xi}) \begin{pmatrix} B - B^* & 0 \\ 0 & B^* - B \end{pmatrix} \begin{pmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{pmatrix} &= (\bar{\xi} \tilde{\xi} - \xi \tilde{\xi}) (B - B^*) \propto (\delta A) \omega_{\text{eff}} \propto L^2 \omega_{\text{eff}}. \end{aligned} \tag{47}$$

The above is, in effect, a restatement of $\nabla_k (L^2 \omega_{\text{eff}}) = 0$ in which $B = \theta_{\text{eff}} - i\omega_{\text{eff}}$ and \top denotes Hermitian transpose.

These invariants reduce to the classical ones in the limit of zero perturbation and therefore there remains no anomaly in QGO. This is the case since we have applied the symmetries present at $O(\alpha^0)$ as a set of constraints and exploited them to find the covariant quantum corrections. Since the effective action, that we started with, has been derived in a gauge invariant manner [1], the results obtained here are also gauge invariaant.

The classical limit of eq. (44) can be compared with that corresponding to the standard (purely general relativistic) Raychaudhuri equation, namely

$$\nabla_k \theta = \omega^2 - \theta^2 - |\sigma|^2 - \frac{1}{2} R_{(1)(1)}. \quad (48)$$

The last RHS term above is negative for all known forms of ponderable matter. For an initially contracting congruence, in the absence of rotation and shear (ρ real and $\sigma = 0$), it leads to the divergence of the expansion parameter ($\theta \rightarrow -\infty$). This is a necessary though not a sufficient condition for proving the singularity theorems [14]. In our case, a real ρ and vanishing σ are equivalent to $\omega_{\text{eff}} = \sigma_{\text{eff}} = 0$. Using (44), we see that during the propagation along the ray, it differs from zero only by $O(\alpha^2)$ [15],

$$\nabla_k \theta = -\theta^2 + \frac{1}{m^2} A_{(3)(4)} \theta^2 + O(\alpha^2). \quad (49)$$

On the other hand, the last RHS term in eq. (48) does not appear in our calculations due to the abreast ray conditions. Even if $R_{(1)(1)} = 0$, however, the correction terms calculated in this paper will contribute and their sign must be taken into account.

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