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Upconversion of whistler waves by gyrating ion beams in a plasma

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Abstract. A gyrating ion beam, with a ring-shaped distribution in velocity, supports negative energy beam modes near the harmonics of beam gyro-frequency. An investigation of the non-linear interaction of high-frequency whistler waves with the negative energy beam cyclotron mode is made. A non-linear dispersion relation is derived for the coupled modes. It is shown that a gyrating ion-beam frequency upconverts the whistler waves separated by harmonics of beam gyro-frequency. The expression for the growth rate of whistler mode waves has been derived. In Case 1, a high-amplitude whistler wave decays into two lower frequency waves, called a low-frequency mode and a side band of frequency lower than that of pump wave. In Case 2 a high-amplitude whistler wave decays into two lower frequency daughter waves, called the low-frequency mode and whistler waves. Generation mechanism of these waves has application in space and laboratory plasmas.

Keywords. Electromagnetic waves; magnetic field; whistler.

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1. Introduction

Parametric excitation is an important non-linear phenomenon in plasmas. Extensive studies of parametric instabilities of electromagnetic waves in plasmas have been reported over the years with applications in laboratory and space plasmas. Parametric instabilities of electromagnetic waves in plasma have received considerable attention in recent years [1–8]. In this process, a high-amplitude pump wave $(\omega_0,\ k_0)$ decays into two lower frequency daughter waves, called a low-frequency mode (ω, k) and a side band $(\omega_1,\ k_1)$ where $\omega, \omega_1 < \omega_0$. In some cases, when the plasma has a source of free energy, parametric upconversion of the pump wave is also possible where the frequency of the side band (ω_1, k_1) is greater than that of the pump wave, i.e., $\omega_1 > \omega_0$.

An important example of parametric upconversion is a free electron laser in which a wiggler magnetic field of zero frequency and wave vector k_0 parametrically excites a laser radiation by extracting energy from the axial beam. Bujarbarua *et al* [9] have shown that the problem of the decay of a lower hybrid wave into a whistler-mode signal and an ion cyclotron in a plasma and results demonstrate that the parametric interaction requires very low amplitude of the pump wave. It is shown that lower hybrid wave can trigger the growth of VLF and ULF noise in a plasma.

In a recent theoretical study, Sharma and Patel [10] have shown that a gyrating electron beam can cause the upconversion of an electromagnetic wave in plasma. A similar physical process of upconversion which may occur in the presence of a gyrating ion beam, with ring distribution in velocity space, are encountered in both laboratories as well as space plasmas. Whistler waves passing through the plasma in the presence of a gyrating ion beam can parametrically upconvert into another wave.

In this paper, we theoretically investigate the non-linear interaction of high-frequency whistler waves with the negative energy beam cyclotron mode resulting in the upconversion of the whistler wave separated by harmonics of the beam gyro-frequency. The whistler waves have wide-ranging applications. They are important in the pitch angle diffusion of energetic electrons, and for the diagnostics of magnetospheric plasma parameters such as electron density in equatorial magnetosphere, duct-localization, and large-scale magnetospheric electric field according to Sazhin *et al* [11] and Singh [12].

The physical mechanism of this process can be understood as follows. Consider the propagation of a whistler wave

$$E_0 = E_0 \exp[-i(\omega_0 t - \vec{k}_0 \cdot \vec{x})],$$

the linear response of electrons is v_0 and the whistler wave decays into a low-frequency mode (ω) and a lower hybrid side band at ($\omega + \omega_0$) frequency. The side band couples to the pump to produce a low-frequency pondermotive force with a component parallel to the ambient magnetic field $\vec{B}_{\rm s}$. This parallel component derives the low-frequency mode that, along with the side band, grows exponentially with time at the expense of the energy of the pump and the free energy of the gyrating ion beam.

In the present paper, we study the non-linear interaction of high-frequency whistler waves with the negative energy beam cyclotron mode. A non-linear dispersion relation is derived and analytic expression for the uniform medium growth rate is obtained. Growth of whistler wave has been increased in response to a set of space plasma parameters.

2. Theoretical considerations

Case 1: *Non-linear dispersion relation*

Consider the propagation of a whistler pump in plasma, in the presence of a static magnetic field $B_S || \hat{z}$. Then

$$\vec{E}_0 = \vec{E}_0 \exp[-i(\omega_0 t - \vec{k}_0 \cdot \vec{z})], \quad (1)$$

where

$$k_0^2 = \frac{\omega_0^2}{c^2}\left(1 - \frac{\omega_{\rm p}^2}{\omega_0(\omega_0 - \omega_{\rm c})}\right)$$

and

$$\omega_{\rm c} = eBs/mc, \quad \omega_{\rm p} = \left(\frac{4\pi n_0 e^2}{m}\right)^{1/2},$$

$$E_{0x} = -iE_{0y}, \quad E_{0z} = 0.$$

This wave produces oscillatory drift of electrons $\vec{v}_0$, solving the equation of motion,

$$\frac{\partial \vec{v}_0}{\partial t} = \frac{-e\vec{E}_0}{m} - \frac{e}{mc}\vec{v}_0 \times \vec{B}_{\rm s}. \tag{2}$$

One obtains

$$v_{0x} = -iv_{0y}, \quad v_{0x} = \frac{eE_{0x}}{im(\omega_0 - \omega_{\rm c})}. \tag{3}$$

The whistler wave ω_0 decays into a low-frequency mode $(\omega, \vec{k})$ and a lower hybrid side band $(\omega_1, \vec{k}_1)$. As the plasma has a source of free energy, we take $\omega_1 = \omega + \omega_0$, i.e., the frequency of the side band $(\omega_1, \vec{k}_1)$ is greater than that of the pump wave. The low-frequency mode is a negative energy beam cyclotron mode.

Let the perturbation of the decay waves be

$$\phi = \phi {\rm e}^{-i(\omega t - \vec{k}\cdot\vec{r})}, \tag{4}$$

$$\phi_1 = \phi_1 {\rm e}^{-i(\omega_1 t - \vec{k}_1\cdot\vec{r})}. \tag{5}$$

The linear response of electrons to the side-band potential can be obtained as

$$\frac{\partial \vec{v}_1}{\partial t} = \frac{-e\vec{E}_1}{m} - \frac{e}{mc}\vec{v}_1 \times \vec{B}_{\rm s} \tag{6}$$

After solving eq. (6), we get

$$v_{1x} = \frac{-e(i\omega_1 E_{1x} + \omega_{\rm c} E_{1y})}{m(\omega_1^2 - \omega_{\rm c}^2)}, \tag{7}$$

$$v_{1y} = \frac{+e(E_{1y}\omega_1 + E_{1x} i\omega_{\rm c})}{mi},$$

$$v_{1z} = \frac{eE_{1z}}{mi\omega_1},$$

where $\vec{E}_1 = -\nabla\phi_1$.

The side-band fields along with the pump (ω_0, k_0) produce a low-frequency pondermotive force

$$\vec{F}_{\rm p} = \frac{-m}{2}[v_0\nabla\vec{v}_1^* + \vec{v}_1\nabla v_0^*] - \frac{e}{2c}[\vec{v}_1 \times B_0^*]. \tag{8}$$

In writing eq. (8) we have used the identity

$$\mathrm{Re}\,A\,\mathrm{Re}\,B = 1/2\mathrm{Re}[AB + AB^*],$$

where Re stands for 'real part of'.

$$F_{\mathrm{p}z} = \frac{e^2}{2m} E_{0x}^* \phi_1 k_{1x} \left[\frac{-ik_{1z}}{\omega_1(\omega_0 - \omega_{\mathrm{c}})} + \frac{k_{0z}}{\omega_0(\omega_1 - \omega_{\mathrm{c}})} \right]. \tag{9}$$

The density perturbation in response to the pondermotive potential is

$$\phi_{\mathrm{p}} = \frac{F_{\mathrm{p}z}}{eik_z}. \tag{10}$$

We solve eq. (10) and get

$$\phi_{\mathrm{p}} = \alpha_2 \phi_1 E_{0x}^*, \tag{11}$$

where

$$\alpha_2 = \frac{e^2 k_{1x}}{2meik_z} \left[\frac{ik_{1z}}{\omega_1(\omega_0 - \omega_{\mathrm{c}})} + \frac{k_{0z}}{\omega_1(\omega_0 - \omega_{\mathrm{c}})} \right].$$

The electron density perturbation n_{e} can be derived from the pondermotive force ϕ_{p} and self-consistent low-frequency potential ϕ,

$$n_{\mathrm{e}} = \frac{k^2}{4\pi e} \chi_{\mathrm{e}}(\phi + \phi_{\mathrm{p}}),$$

where

$$\phi = \phi \mathrm{e}^{-i(\omega t - \vec{k}\cdot z)}. \tag{12}$$

The pondermotive force on the ions is down by the electron to ion mass ratio; hence, their density can be taken to be linear.

$$n_{\mathrm{i}} = \frac{-k^2 \chi_{\mathrm{i}} \phi}{4\pi e},$$

where we have assumed ions to have single charge, using n_{e} and n_1 in the Poisson's equation,

$$\nabla^2 \phi = 4\pi e(n_{\mathrm{e}} - n_{\mathrm{i}}). \tag{13}$$

We may get

$$\varepsilon \phi = -\chi_{\mathrm{e}} \phi_{\mathrm{p}}. \tag{14}$$

The low-frequency density perturbation n couples to the oscillatory velocity $\vec{v}_0$ to produce a non-linear side-band current $(n_{\mathrm{e}} \cdot \vec{v}_0)/2$ and a density perturbation n_1^{NL}.

$$n_1^{\mathrm{NL}} = \frac{1}{2} \frac{n_{\mathrm{e}} \vec{k}_1 \cdot \vec{v}_0}{\omega_1},$$

$$n_1 = \frac{n_0^0 (k_{1x})^2 e\phi_1}{m(\omega_1^2 - \omega_c^2)} - \frac{(k_{1z})^2 e\phi_1 n_0^0}{m(\omega_1^2)} - \frac{1}{2}\frac{ik^2}{4\pi e}\frac{\chi_e(\phi+\phi_p)}{m(\omega_0-\omega_c)\omega_1} eE_{0x}k_{1x}. \quad (15)$$

Using eq. (15) in Poisson's equations, one obtains

$$\phi_1(1+\chi_{e1}) = \frac{ik^2}{2k_1^2}\frac{\chi_e(\phi)eE_{0x}k_{1x}n_1}{m(\omega_0-\omega_c)\omega_1}, \quad (16)$$

where

$$\chi_{e1} = -\left\{ \frac{\omega_p^2}{\omega_1^2 - \omega_c^2}\frac{k_{1x}^2}{k_1^2} + \frac{\omega_p^2}{\omega_1^2}\frac{k_{1z}^2}{k_1^2} \right\}$$

and $\varepsilon_1 = 1 + \chi_{e1}$, $\varepsilon_1\phi_1 = \alpha_1\phi$.

$$\alpha_1 = \frac{ik^2}{2k_1^2}\frac{\chi_e e E_{0x}k_{1x}n_1}{m\omega_1(\omega_0-\omega_c)}. \quad (17)$$

Using eqs (11), (14) and (17) leads to the non-linear dispersion relation as

$$\varepsilon\varepsilon_1 = \alpha_1\alpha_2 E_{0x}^* E_{0x}\chi_e = \mu. \quad (18)$$

The low-frequency and the side-bands are thus non-linearly coupled. Here

$$\varepsilon = 1 + \chi_e + \chi_i. \quad (19)$$

The dispersion relation for the low-frequency mode is given by

$$\varepsilon = 1 + \frac{\omega_{pi}^2}{k^2 c_s^2} + \frac{2\omega_{pi}^2}{k^2 v_i^2}(1-\eta)\left[1 + \frac{\omega}{k_z v_i}\sum Z\frac{\omega - n\omega_{ci}}{k_z v_i} I_n e^{-b_i}\right]$$
$$+\eta\frac{\omega_{pi}^2}{\omega_{cb}^2}\left(\frac{k_\perp^2}{k^2}\sum_n \frac{J_{n-1}^2 - J_{n+1}^2}{2\left(\frac{\omega}{\omega_{cb}} - n\right)} + \frac{k_z^2}{k^2}\sum_n \frac{J_n^2}{\left(\frac{\omega}{\omega_{cb}} - n\right)^2}\right). \quad (20)$$

The last term represents the beam contribution with $\eta = 1$. Z is the usual Fried and Conte [13] plasma dispersion function, where I_n is the nth order modified Bessel function with argument b_i, J_n is the nth order Bessel function of argument $(K_\perp V_{\perp b})/\omega_{cb}$, and $b_i = (k_\perp^2\rho_i^2)/2$ with $\rho_i = v_i/\omega_{ci}$ and $c_s = \sqrt{(T_e/m_i)}$. In writing eq. (20) we have assumed $\omega \ll k_z v_e$. Here v_i and v_e are the thermal velocities of ions and electrons, respectively. In this case electrons can effectively follow potential variations in the z direction. Also their response to non-linear forces, arising through the interaction of high-frequency mode is quite strong.

We express ω as $\omega_r + i\gamma$, where ω_r is the resonant frequency at which $\varepsilon = 0$ and $\varepsilon_1 = 0$. By Taylor expanding ε_1 around $\omega_1 = \omega_0 + i\gamma$ and when $\varepsilon(\omega, k)$ is expanded around a real frequency then $\varepsilon(\omega, k)$ is not zero; instead

$$\varepsilon(\omega,k) = \varepsilon(\omega_r,k) + i\gamma\frac{\partial\varepsilon}{\partial\omega}. \quad (21)$$

We obtain from eq. (18)

$$\gamma^2 - \frac{i\varepsilon(\omega_{\mathrm{r}}, k)}{\partial\varepsilon/\partial\omega}\gamma + \frac{\mu}{(\partial\varepsilon/\partial\omega)(\partial\varepsilon_1/\partial\omega_1)} = 0. \tag{22}$$

Equation (22) has the roots,

$$\gamma = \gamma_{\mathrm{r}} + i\gamma_{\mathrm{i}}.$$

Here γ is a complex quantity. The imaginary part γ_{i} gives the phase shift, whereas the real part γ_{r} gives the growth rate. Thus the growth rate is given as

$$\gamma_{\mathrm{r}}^2 = \frac{-\mu}{(\partial\varepsilon/\partial\omega)(\partial\varepsilon_1/\partial\omega_1)}. \tag{23}$$

From eqs (11), (16)–(19) we get

$$\begin{aligned} \varepsilon_1 &= 1 - \left(\frac{\omega_{\mathrm{p}}^2}{\omega_1 - \omega_{\mathrm{c}}^2} \frac{k_{1x}^2}{k_1^2} + \frac{\omega_{\mathrm{p}}^2}{\omega_1^2} \frac{k_{1z}^2}{k_1^2} \right), \\ \frac{\partial\varepsilon_1}{\partial\omega} &= 0 + 0 + 2\frac{\omega_{\mathrm{p}}^2}{\omega_1^3}\frac{k_{1z}^2}{k_1^2}. \end{aligned} \tag{24}$$

From eq. (20), we get

$$\frac{\partial\varepsilon}{\partial\omega} = \frac{\eta\omega_{\mathrm{pi}}^2}{\omega_{\mathrm{cb}}^2}\frac{J_n^2}{\omega_{\mathrm{cb}}}\frac{k_z^2}{k^2}\frac{1}{((\omega/\omega_{\mathrm{cb}}) - n)^3}. \tag{25}$$

Substituting eqs (24) and (25) in eq. (23), we get the expression for the growth rate as

$$\begin{aligned} \gamma_{\mathrm{r}}^2 &= \frac{-\mu}{\frac{\partial\varepsilon}{\partial\omega}\frac{\partial\varepsilon_1}{\partial\omega_1}} \\ &= \frac{\left[\frac{ik^2}{2k_1^2}\frac{\chi_{\mathrm{e}} eE_{0x}k_{1x}}{m(\omega_0-\omega_{\mathrm{c}})\omega_1}\right]\left[\frac{ek_{1x}}{2mk_z}\left(\frac{ik_{1z}}{\omega_1(\omega_0-\omega_{\mathrm{c}})} + \frac{k_{0z}}{\omega_0(\omega_1-\omega_{\mathrm{c}})}\right)E_{0x}E_{0x}^*\chi_{\mathrm{e}}\right]}{2\eta\frac{\omega_{\mathrm{pi}}^2}{\omega_{\mathrm{cb}}^2}\frac{J_n^2}{\omega_{\mathrm{cb}}}\frac{k_z^2}{k^2}\frac{1}{((\omega/\omega_{\mathrm{cb}})-n)^3}\left(\frac{2\omega_{\mathrm{p}}^2}{\omega_1^3}\frac{k_{1z}^2}{k_1^2}\right)}, \end{aligned} \tag{26}$$

$$\left(\frac{\gamma_{\mathrm{r}}}{\omega_{\mathrm{cb}}}\right)^2 = \frac{\frac{1}{4}\left[\frac{k^2}{k_1^2}\frac{\chi_{\mathrm{e}}^2 eE_{0x}}{m\omega_{\mathrm{c}}\left(\frac{\omega_0}{\omega_{\mathrm{c}}}-1\right)}\frac{eE_{0x}}{m\omega_{\mathrm{c}}\left(\frac{\omega_1}{\omega_{\mathrm{c}}}-1\right)}\frac{k_{0z}(k_{1x})^2\omega_1}{k_z\omega_0(\omega_{\mathrm{cb}})^2\omega_1}\right]}{2\eta\frac{\omega_{\mathrm{pi}}^2}{\omega_{\mathrm{cb}}^2}\frac{J_n^2}{\omega_{\mathrm{cb}}}\frac{k_z^2}{k^2}\frac{2}{((\omega/\omega_{\mathrm{cb}})-n)^3}\frac{\omega_{\mathrm{p}}^2}{\omega_1^2}\frac{k_{1z}^2}{k_1^2}}, \tag{26a}$$

$$\left(\frac{\gamma_{\mathrm{r}}}{\omega_{\mathrm{cb}}}\right)^2 = \frac{\frac{1}{8}\left[\frac{k^2}{k_1^2}\frac{\chi_{\mathrm{e}}^2 eE_{0x}}{m\omega_{\mathrm{c}}((\omega_0/\omega_{\mathrm{c}})-1)}\frac{eE_{0x}}{m\omega_{\mathrm{c}}(\omega_1/(\omega_{\mathrm{c}}-1))}\frac{k_{0z}(k_{1x})^2\omega_1}{k_z\omega_0(\omega_{\mathrm{cb}})^2\omega_1}\right]}{2\eta\frac{\omega_{\mathrm{pi}}^2}{\omega_{\mathrm{cb}}^2}\frac{J_n^2}{\omega_{\mathrm{cb}}}\frac{k_z^2}{k^2}\frac{2}{((\omega/\omega_{\mathrm{cb}})-n)^3}\frac{\omega_p^2}{\omega_1^2}\frac{k_{1z}^2}{k_1^2}}, \tag{27}$$

where ω_1 is the frequency of lower hybrid side-band.

Case 2

We consider warm background plasma, embedded in a uniform magnetic field $B_{\rm s}||\hat{z}$ in which both electrons and ions have Maxwellian distributions with two different temperatures with ring-shape distribution in velocity space. The latter is called a gyrating beam. It supports a negative energy mode around the harmonics of beam gyro-frequency and can give rise to frequency upconversion of the whistler waves separated by harmonics of the beam gyro-frequency.

But in Case 2 a high-amplitude whistler wave decays into two lower frequency daughter waves, called the low-frequency mode and a whistler wave. However, when the plasma has a source of free energy, upconversion of the pump wave is also possible, where the frequency of the whistler wave is greater than that of the pump wave.

Consider the propagation of a whistler wave in plasma in the presence of a static magnetic field $B_{\rm s}||\hat{z}$ given by

$$\begin{aligned} E_0 &= A_0(\hat{x}+i\hat{y})\exp -i(\omega_0 t - \vec{k}_0 z), \\ B_0 &= \frac{c\vec{k}_0 \times \vec{E}_0}{\omega_0}, \end{aligned} \tag{28}$$

where

$$k_0^2 = \frac{\omega_0^2}{c^2}\left(1 - \frac{\omega_{\rm p}^2}{\omega_0(\omega_0-\omega_{\rm c})}\right)$$

with $k_0 = k_{0x}\hat{x} + k_{0z}\hat{z}$.

The linear response of electrons to this wave can be written as

$$\frac{\partial \vec{v}_0}{\partial t} = -\frac{eE_0}{m} - \frac{e\vec{v}_0 \times \vec{B}}{mc}. \tag{29}$$

For the pump whistler wave

$$\begin{aligned} E_{0y} &= iE_{0x}, \\ v_{0x} &= \frac{eE_{0x}}{im(\omega_0-\omega_{\rm c})}, \\ v_{0y} &= iv_{0x}, \\ \omega_{\rm r} &\ll \omega_{\rm c}, \quad E_{0z} \cong 0, \quad E_{1z} \cong 0. \end{aligned} \tag{30}$$

Near the resonance, the wave attains high amplitude and the whistler wave decays into a low-frequency mode (ω, k) and a whistler wave $(\omega_1,\ k_1)$, when the plasma has a source of free energy, the phase match conditions for frequency upconversion demand that

$$\omega_1 = \omega + \omega_0, \quad \vec{k}_1 = \vec{k} + \vec{k}_0. \tag{31}$$

Now

$$\vec{E}_1 = \vec{E}_1 \exp -i(\omega_1 t - \vec{k}_1 r),$$
$$\vec{B}_1 = \frac{c(\vec{k}_1 \times \vec{E}_1)}{\omega_1}. \tag{32}$$

The linear response of electrons to this wave can be written as

$$v_{1x} = \frac{-e(i\omega_1 E_{1x} + \omega_{\rm c} E_{1y})}{m(\omega_1^2 - \omega_{\rm c}^2)},$$
$$v_{1y} = \frac{e(E_{1y}\omega_1 + iE_{1x}\omega_{\rm c})}{mi(\omega_1^2 - \omega_{\rm c}^2)},$$
$$v_{1z} = \frac{eE_{1z}}{mi\omega_1}. \tag{33}$$

The pondermotive and self-consistent potentials produce an electron density perturbation which is given by

$$n_{\rm e} = \frac{k^2}{4\pi e}\chi_{\rm e}(\phi + \phi_{\rm p}), \tag{34}$$

where $\chi_{\rm e}$ is the electron susceptibility at ω, k. For $\omega \ll kv_{\rm th}$ ($v_{\rm th}$ being the electron thermal speed)

$$\chi_{\rm e} = \frac{2\omega_{\rm p}^2}{v_{\rm th}^2 k^2} = \frac{\omega_{\rm p}^2}{k^2 c_{\rm s}^2}. \tag{35}$$

The pondermotive force on the ions is down by the electron to ion mass ratio, and hence the ion density perturbation can be written as

$$n_{\rm i} = \frac{k^2}{4\pi e}\chi_{\rm i}\phi. \tag{36}$$

Using $n_{\rm e}$ and $n_{\rm i}$ in the Poisson's equation

$$\nabla^2\phi = 4\pi e(n_{\rm e} - n_{\rm i}),$$

we obtain

$$\varepsilon\phi = -\chi_{\rm e}\phi_{\rm p}, \quad \text{where} \quad \varepsilon = 1 + \chi_{\rm e} + \chi_{\rm i},$$
$$n_{\rm e} = \frac{k^2}{4\pi e}(\chi_{\rm e} - \varepsilon)\phi, \tag{37}$$

where ε is small.

Side-band response

The low-frequency density perturbation n couples with the oscillatory velocity v_0 to produce a non-linear side-band current $(nv_0)/2$ and a current density

$$\vec{J}_1^{\rm NL} = \frac{-en_{\rm e}\vec{v}_0}{2}. \tag{38}$$

From eq. (37)

$$\vec{J}_1^{\rm NL} = \frac{-k^2}{8\pi}\chi_{\rm e}\phi\vec{v}_0. \tag{39}$$

Wave equation for the side-band whistler wave is

$$\nabla \times \vec{E}_1 = \frac{-1}{c}\frac{\partial \vec{B}_1}{\partial t}, \tag{40}$$

$$\nabla \times \vec{H}_1 = \frac{4\pi \vec{J}_1}{c} + \frac{1}{c}\frac{\partial \vec{D}_1}{\partial t}, \tag{41}$$

where

$$\text{Curl } \vec{E} = \frac{-1}{c}\frac{\partial \vec{B}}{\partial t},$$

$$\text{Curl } \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c}\vec{J}.$$

We know that $\vec{D} = \varepsilon\vec{E}, \vec{J} = \sigma\vec{E}, \vec{B} = \mu\vec{H}$,

$$i\vec{k}_1 \times \vec{E}_1 = i\frac{\omega_1\vec{H}_1}{c}, \tag{42}$$

$$i\vec{k}_1 \times \vec{H}_1 = \frac{4\pi}{c}\underline{\underline{\sigma_1}}E_1 - i\frac{\omega_1}{c}\vec{E}_1 + \frac{4\pi}{c}\vec{J}_1^{\rm NL}. \tag{43}$$

Taking $\vec{k}_1\times$ of eq. (42)

$$\vec{k}_1 \times (\vec{k}_1 \times \vec{E}_1) = \frac{\omega_1}{c}\vec{k}_1 \times \vec{H}_1, \tag{44}$$

where

$$\underline{\underline{\varepsilon_1}} = \underline{\underline{I}} + \frac{4\pi i\underline{\underline{\sigma_1}}}{\omega_1}$$

and I is a unit tensor

$$I = \left(\frac{\omega_1^2}{c^2}\underline{\underline{\varepsilon_1}}\cdot\vec{E}_1 - k_1^2\,\underline{\underline{I}}\,\vec{E}_1 + \vec{k}_1\cdot\vec{k}_1\vec{E}_1\right) = \frac{4\pi}{ic^2}\omega_1 J_1^{\rm NL} \tag{45}$$

$$= \frac{c^2}{\omega_1^2}\underline{\underline{D_1}}\cdot\vec{E}_1 = \frac{-4\pi i\vec{J}_1^{\rm NL}}{\omega_1} = \underline{\underline{D_2}}\vec{E}_1, \tag{46}$$

where

$$\underline{\underline{D_1}} = \frac{\omega_1^2}{c^2}\underline{\underline{\varepsilon_1}} - \underline{\underline{I}}k_1^2 + \vec{k}_1\cdot\vec{k}_1.$$

Here

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{1xx} & \varepsilon_{1xy} & \varepsilon_{1xz} \\ \varepsilon_{1yx} & \varepsilon_{1yy} & \varepsilon_{1yz} \\ \varepsilon_{1zx} & \varepsilon_{1zy} & \varepsilon_{1zz} \end{pmatrix},$$

where $\underline{\underline{\varepsilon_1}}$ is a dielectric tensor.

$Ik_1 = I(k_{1x}k_{1y}k_{1z})$, where I is a unit tensor

$$\begin{pmatrix} k_{1x} & 0 & 0 \\ 0 & k_{1y} & 0 \\ 0 & 0 & k_{1z} \end{pmatrix}$$

$$k_1^2 = k_{1x}^2 + k_{1y}^2 + k_{1z}^2 = k_{1x}^2 + k_{1z}^2,$$

$$\vec{k}_1 \cdot \vec{k}_1 = \begin{pmatrix} k_{1x}k_{1x} & k_{1x}k_{1y} & k_{1x}k_{1z} \\ k_{1y}k_{1x} & k_{1y}k_{1y} & k_{1y}k_{1z} \\ k_{1z}k_{1x} & k_{1z}k_{1y} & k_{1z}k_{1z} \end{pmatrix}.$$

$K_{1y} = 0$ because it is only in x and z directions. Therefore,

$$\underline{\underline{D_2}} = \underline{\underline{D_1}}\frac{c^2}{\omega_1^2} = \begin{pmatrix} \varepsilon_{1xx} - \frac{c^2k_{1z}^2}{\omega_1^2} & \varepsilon_{1xy} & k_{1x}k_{1z} \\ -\varepsilon_{1xy} & \varepsilon_{1yy} - \frac{c^2k_1^2}{\omega_1^2} & 0 \\ k_{1x}k_{1z} & 0 & \varepsilon_{1zz} - \frac{c^2k_{1x}^2}{\omega_1^2} \end{pmatrix}. \tag{47}$$

Since D_{1zz} is very large, ε_{1zz} is very large when compared to other elements.

For $\omega_1 \ll \omega_\mathrm{c}$, $\varepsilon_{1xx} < \varepsilon_{1xy}$,

$$\varepsilon_{1xx} = 1 - \frac{\omega_\mathrm{p}^2}{\omega_1^2 - \omega_\mathrm{c}^2} = \varepsilon_{1yy},$$

$$\varepsilon_{1xy} = -\varepsilon_{1yx} = -i\frac{\omega_\mathrm{c}}{\omega_1}\frac{\omega_\mathrm{p}^2}{\omega_1^2 - \omega_\mathrm{c}^2},$$

$$\varepsilon_{1zz} = 1 - \frac{\omega_\mathrm{p}^2}{\omega_1^2} = \frac{-\omega_\mathrm{p}^2}{\omega_1^2},$$

$$\varepsilon_{1xz} = \varepsilon_{1zx}. \tag{48}$$

$$|\underline{\underline{D_2}}| = \frac{-\omega_\mathrm{p}^2}{\omega_1^2}\left[\frac{c^4k_{1z}^2k_1^2}{\omega_1^4} - \frac{\omega_\mathrm{c}^2}{\omega_1^2}\frac{\omega_\mathrm{p}^2}{\omega_\mathrm{c}^2}\right], \tag{49}$$

$$= \frac{-\omega_\mathrm{p}^2}{\omega_1^2}\frac{2\omega_\mathrm{p}^2}{\omega_1\omega_\mathrm{c}}\frac{c^2}{\omega_1^2}\left(k_1k_{1z} - \frac{\omega_1\omega_\mathrm{p}^2}{c^2\omega_\mathrm{c}}\right), \tag{50}$$

if $\omega_0 \ll \omega_\mathrm{c}$; $\omega_1 \ll \omega_\mathrm{c}$.

In the presence of the pump and the whistler side-band, electrons experience a pondermotive force with a component parallel to the ambient magnetic field, that is,

$$F_{\mathrm{p}z} = eik_z\phi_{\mathrm{p}}, \tag{51}$$

$$F_{\mathrm{p}} = \frac{-m}{2}(\vec{v}_0^*\nabla\vec{v}_1 + \vec{v}_1\nabla v_0^*) - \frac{e}{2c}(\vec{v}_0^* \times \vec{B}_1 + \vec{v}_1 \times \vec{B}_0^*). \tag{52}$$

For $E_{1z} \cong 0$, $v_{1z} \cong 0$, $v_{0z} = 0$,

$$\phi_{\mathrm{p}} = \frac{-1}{2ik_z}\left(\frac{k_{1z}}{\omega_1}\vec{v}_0^* \cdot \vec{E}_1 + \frac{k_{0z}}{\omega_0}\vec{v}_1 \cdot \vec{E}_0^*\right). \tag{53}$$

We know that (from eq. (19))

$$\underline{\underline{D_2}} \cdot \vec{E}_1 = \underline{\underline{D_1}} \cdot \vec{E}_1 \cdot \frac{c^2}{\omega_1^2} = \frac{-i4\vec{J}_1^{\mathrm{NL}}}{\omega_1}. \tag{54}$$

If $\omega_0 \ll \omega_{\mathrm{c}}$ then $E_{1z} \cong 0$. In that case write down the x and y components of the equation

$$D_{2xx}E_{1x} + D_{2xy}E_{1y} = \frac{4\pi i}{\omega_1}\left(\frac{1}{2}n_{\mathrm{e}}v_{0x}\right), \tag{55}$$

$$D_{2yx}E_{1x} + D_{2yy}E_{1y} = \frac{4\pi i}{\omega_1}\left(\frac{1}{2}n_{\mathrm{e}}v_{0y}\right). \tag{56}$$

These equations can be solved to obtain E_{1x}, E_{1y} in terms of ϕ.

$$E_{1x} = \frac{i4\pi}{\omega_1}\frac{1}{2}\frac{k^2}{4\pi}\frac{(\chi_{\mathrm{e}}\phi)(v_{0x}D_{2yy} - v_{0y}D_{2xy})}{(D_{2xx}D_{2yy} - D_{2yx}D_{2xy})}, \tag{57}$$

$$E_{1y} = \frac{i4\pi}{\omega_1}\frac{1}{2}\frac{k^2}{4\pi}\frac{(\chi_{\mathrm{e}}\phi)(v_{0y}D_{2xx} - v_{0x}D_{2yx})}{(D_{2xx}D_{2yy} - D_{2yx}D_{2xy})}. \tag{58}$$

Let

$$E_{1x} = b_1\phi, \quad E_{2y} = b_2\phi,$$
$$v_{0y} = iv_{0x}, \quad E_{0y} = iE_{0x},$$
$$v_{0x}^*\,E_{1x} + v_{0y}^*\,E_{1y} = \vec{v}_0^*\,E_1, \tag{59}$$

$$v_{1x}^*\,E_{0x} + v_{1y}^*\,E_{0y} = \vec{v}_1^*\,E_0. \tag{60}$$

Substitute the values of v_{1x}, v_{1y} in terms of E_{1x}, E_{1y} and then $E_{1x} = b_1\phi$ and $E_{1y} = b_2\phi$. This will give ϕ_{p} in terms of ϕ.

$$\phi_{\mathrm{p}} = \frac{-1}{2ik_z}\left[\frac{k_{1z}}{\omega_1}v_{0x}^*(E_{1x} - iE_{1y}) + \frac{k_{0z}E_{0x}^*}{\omega_0}\left(\frac{-e(\omega_1 + \omega_{\mathrm{c}})}{m(\omega_1^2 - \omega_{\mathrm{c}}^2)}\right)(E_{1y} + iE_{1x})\right]. \tag{61}$$

From eq. (14)

$$\varepsilon\phi = -\chi_{\rm e}\phi_{\rm p},$$
$$\varepsilon\phi = \chi_{\rm e}\frac{\phi}{2ik_z}\left[\frac{k_{1z}}{\omega_1}v_{0x}^*(b_1 - ib_2) - \frac{ek_{0z}E_{0x}^*(b_2+ib_1)}{m\omega_0(\omega_1-\omega_{\rm c})}\right],$$
$$\varepsilon = \frac{\chi_{\rm e}}{2ik_z}\left[\frac{k_{1z}}{\omega_1}v_{0x}^*(b_1 - ib_2) - \frac{ek_{0z}E_{0x}^*(b_2+ib_1)}{m\omega_0(\omega_1-\omega_{\rm c})}\right]. \tag{62}$$

Here

$$v_{0x} = \frac{eE_{0x}}{im(\omega_0-\omega_{\rm c})}, \quad v_{0x}^* = \frac{eE_{0x}^*}{-im(\omega_0-\omega_{\rm c})}, \tag{63}$$

$$\varepsilon = (b_1 - ib_2)v_{0x}^*\left(\frac{k_{1z}}{\omega_1} - \frac{k_{0z}}{\omega_0}\right)\frac{\chi_{\rm e}}{2ik_z}. \tag{64}$$

After putting the values of b_1 and b_2,

$$\varepsilon = \frac{(\chi_{\rm e})^2k^2v_{0x}^*}{4k_z\omega_1}\frac{\left(\frac{k_{1z}}{\omega_1}-\frac{k_{0z}}{\omega_0}\right)(v_{0x}D_{2yy} - v_{0y}D_{2xy} - iv_{0y}D_{2xx} - iv_{0x}D_{2yx})}{(D_{2xx}D_{2yy} - D_{2yx}D_{2xy})} \tag{65}$$

$$\mu = \varepsilon(D_{2xx}D_{2yy} - D_{2yx}D_{2xy}) = \frac{(\chi_{\rm e})^2k^2v_{0x}^*}{4k_z\omega_1}\left(\frac{k_{1z}}{\omega_1}-\frac{k_{0z}}{\omega_0}\right)$$
$$\times v_{0x}(D_{2yy} - iD_{2xy} + D_{2xx} - iD_{2yx}). \tag{66}$$

Substituting the values of D_{2xx}, D_{2xy}, D_{1yy}, D_{2yx} in eq. (66), we get non-linear dispersion relation

$$\varepsilon(D_{2xx}D_{2yy} - D_{2yx}D_{2xy}) = \varepsilon F \tag{67}$$

$$\mu = \frac{(\chi_{\rm e})^2k^2v_{0x}^*v_{0x}}{4k_z\omega_1}\left\{2\left(1-\frac{\omega_{\rm p}^2}{\omega_1^2-\omega_{\rm c}^2}\right) - \frac{c^2k_1^2}{\omega_1^2} - \frac{c^2k_{1x}^2}{\omega_1^2}\right\}\left(\frac{k_{1z}}{\omega_1}-\frac{k_{0z}}{\omega_0}\right). \tag{68}$$

The expression for growth rate is

$$\gamma_{\rm r}^2 = \frac{-\mu}{(\partial\varepsilon/\partial\omega)(\partial F/\partial\omega_1)}. \tag{69}$$

As in Case 1, the dispersion relation for the low-frequency mode is

$$\varepsilon = 1 + \chi_{\rm e} + \chi_{\rm i},$$
$$\varepsilon = 1 + \frac{\omega_{\rm pi}^2}{k^2c_{\rm s}^2} + \frac{2\omega_{\rm pi}^2}{k^2v_{\rm i}^2}(1-\eta)\left[1 + \frac{\omega}{k_zv_{\rm i}}\sum_n z\left(\frac{\omega - n\omega_{\rm ci}}{k_2v_{\rm i}}\right)\ln {\rm e}^{-{\rm bi}}\right]$$
$$+\eta\frac{\omega_{\rm pi}^2}{\omega_{\rm cb}^2}\left(\frac{k_\perp^2}{k^2}\sum_n\frac{J_{n-1}^2 - J_{n+1}^2}{2\left(\frac{\omega}{\omega_{\rm cb}}-n\right)} + \frac{k_z^2}{k^2}\sum_n\frac{J_n^2}{\left(\frac{\omega}{\omega_{\rm cb}}-n\right)^2}\right),$$

$$\frac{\partial\varepsilon}{\partial\omega} = \frac{\eta\omega_{\rm pi}^2}{\omega_{\rm cb}^2}\frac{J_n^2}{\omega_{\rm cb}}\frac{k_z^2}{k^2}\frac{(-2)}{\left(\frac{\omega}{\omega_{\rm cb}} - n\right)^3}, \tag{70}$$

$$F = (D_{2xx}D_{2yy} - D_{2yx}D_{2xy})$$
$$= \left(\frac{2\omega_{\rm p}^2}{\omega_1\omega_{\rm c}}\frac{c^2}{\omega_1^2}\left(k_1k_{1z} - \frac{\omega_1\omega_{\rm p}}{c^2\omega_{\rm c}}\right)\right),$$

$$\frac{\partial F}{\partial\omega_1} = \left(\frac{\omega_{\rm p}^2}{c^2\omega_{\rm c}}\right)\left(\frac{4\omega_{\rm p}^2c^2}{\omega_1^3\omega_{\rm c}}\right). \tag{71}$$

Here the final expression is given by

$$\gamma_{\rm r}^2 = \frac{\frac{-(\chi_{\rm e})^2k^2v_{0x}^*v_{0x}}{2k_z\omega_1}\left\{2\left(1 - \frac{\omega_{\rm p}^2}{\omega_1^2-\omega_{\rm c}^2}\right) - \frac{c^2k_1^2}{\omega_1^2} - \frac{c^2k_{1x}^2}{\omega_1^2}\right\}\left(\frac{k_{1z}}{\omega_1} - \frac{k_{0z}}{\omega_0}\right)}{\frac{\eta\omega_{\rm pi}^2}{\omega_{\rm cb}^2}\frac{J_n^2}{\omega_{\rm cb}}\frac{k_z^2}{k^2}\frac{(-2)}{\left(\frac{\omega}{\omega_{\rm cb}}-n\right)^3}\left(\frac{\omega_{\rm p}^2}{c^2\omega_{\rm c}}\right)\frac{4\omega_{\rm p}^2c^2}{\omega_1^3\omega_{\rm c}}}. \tag{72}$$

The expression for the normalized growth is given as

$$\left(\frac{\gamma_{\rm r}}{\omega_{\rm cb}}\right)^2 = \frac{\frac{-(\chi_{\rm e})^2k^2v_{0x}^*v_{0x}}{8k_z\omega_1}\left\{2\left(1 - \frac{\omega_{\rm p}^2}{\omega_1^2-\omega_{\rm c}^2}\right) - \frac{c^2k_1^2}{\omega_1^2} - \frac{c^2k_{1x}^2}{\omega_1^2}\right\}\left(\frac{k_{1z}}{\omega_1} - \frac{k_{0z}}{\omega_0}\right)}{\frac{\eta\omega_{\rm pi}^2}{\omega_{\rm cb}^2}\frac{J_n^2}{\omega_{\rm cb}}\frac{k_z^2}{k^2}\frac{(\omega_{\rm cb})^2}{\left(\frac{\omega}{\omega_{\rm cb}}-n\right)^3}\left(\frac{\omega_{\rm p}^2}{c^2\omega_{\rm c}}\right)\frac{4\omega_{\rm p}^2c^2}{\omega_1^3\omega_{\rm c}}}. \tag{73}$$

3. Results and discussion

Case 1

In figure 1 we have shown that variation of $\gamma_{\rm r}$ normalized to beam gyro-frequency ($\omega_{\rm cb}$) is a function of $[(\omega/\omega_{\rm cb}) - n]$ for a typical set of parameters viz., electron density $n_0^0 = 1$ cm^{-3}, plasma ion mass ($m_{\rm i}$), $m_{\rm i}/m = 2000$ (hydrogen plasma), ion beam mass ($m_{\rm b}$), $m_{\rm b}/m_{\rm i} = 4$ (gyrating helium ions), where m is the mass of electron, plasma frequency, $\omega_{\rm p} = 5\times10^4$ rad/s, cyclotron frequency, $\omega_{\rm c} = \omega_{\rm p}/10$ or $\omega_{\rm p}/40$, gyro-frequency of beam ions, $\omega_{\rm cb} = \omega_{\rm c}(m/m_{\rm b}) \sim \omega_{\rm c}(1/8000)$, pump wave amplitude E_{0x} related with v_{0x} is given by, $v_{0x} = eE_{0x}/(m(\omega_0 - \omega_{\rm c})) = 10^5$ or 10^3, plasma electron temperature, $T_{\rm e}$ ∼10 eV and pump wave frequency $\omega_0 = 0.5\omega_{\rm c}$.

It should be noted that the mode has negative energy only in the vicinity of $\omega_{\rm cb}$. This fact is also obvious from the graph. When ω is exactly equal to $\omega_{\rm cb}$, the growth rate is zero. As the difference $(\omega - \omega_{\rm cb})$ increases, the growth rate also increases rapidly, reaching a maximum when ω is very nearly equal to $1.08\omega_{\rm cb}$. Thus a whistler wave, as a result of the non-linear interaction with the negative energy beam cyclotron mode, is converted into another lower hybrid wave with a frequency higher than that of the pump wave. The process is therefore called frequency upconversion. The separation in frequency of the pump and the side-band waves is equal to the harmonics of the beam gyro-frequency. Thus by knowing the separation in frequency of the pump and the side-band waves, this process of upconversion can be used as a diagnostic tool for detecting ion beams in a plasma.

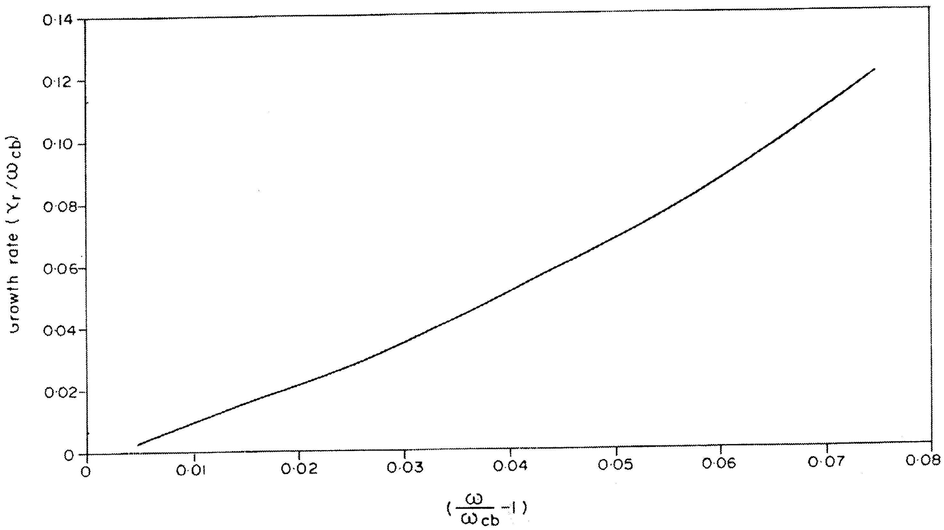

Figure 1. Variation of growth rate (γ_r/ω_{cb}) of whistler wave normalized to beam gyro-frequency as a function of $\{(\omega/\omega_{cb}) - 1\}$.

Case 2

In figure 2 we have shown that variation of (γ_r/ω_{cb}), normalized to beam gyro-frequency, is a function of $\{(\omega/\omega_{cb}) - 1\}$ for a typical set of parameters as given in Case 1.

Figure 2 shows that as the difference $(\omega - \omega_{cb})$ increases, the growth rate also increases rapidly, reaching a maximum when ω is equal to $1.8\omega_{cb}$. Thus a whistler wave, as a result of the non-linear interaction with the negative energy beam cyclotron mode, is converted into another whistler wave with a frequency higher than that of the pump wave.

A gyrating ion beam supports negative energy modes near the harmonics of beam cyclotron frequency. A whistler wave passing through such a beam parametrically upconverts into high-frequency whistler modes separated from the pump frequency by the beam cyclotron harmonics.

4. Conclusion

In this paper we examine the possibility of parametric instability of the whistler waves. If a whistler wave is excited in such a system, then it resonantly decays into a negative energy beam cyclotron mode and a frequency upshifted upper side band. This process thus upshifts the whistler spectrum. Conversely, a decay wave spectrum consisting of whistler waves upshifted and separated by beam gyro-frequency is a positive indication of the presence of a gyrating ion beam in the plasma. This process of parametric upconversion of whistler waves thus, could provide a potential diagnostic tool for detecting gyrating ion beams in a plasma. Parametric instabilities of whistler waves in plasmas have received considerable attention in recent

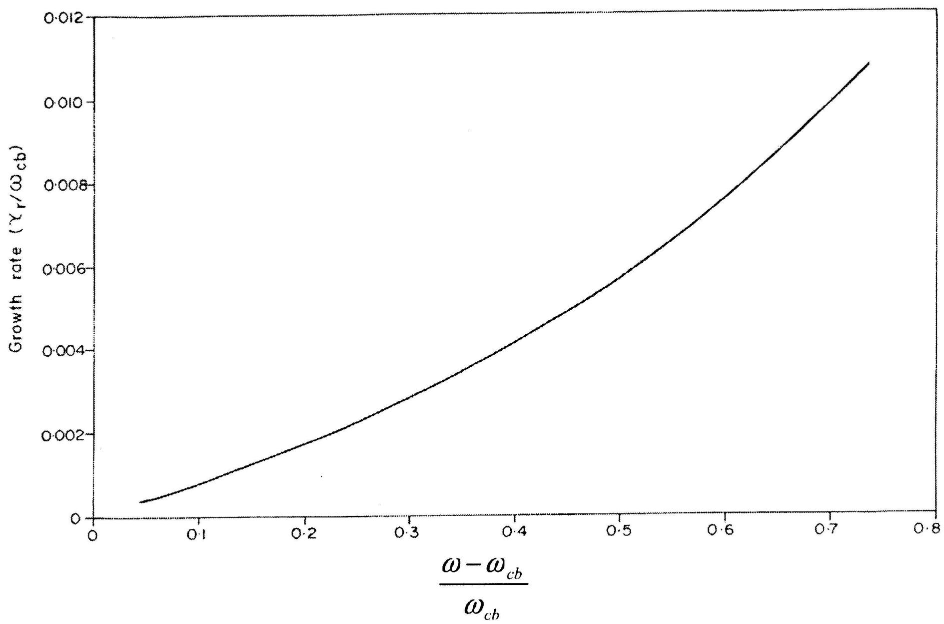

Figure 2. Variation of growth rate $(\gamma_\mathrm{r}/\omega_\mathrm{cb})$ of whistler wave normalized to beam gyro-frequency as a function of $((\omega-\omega_\mathrm{cb})/\omega_\mathrm{cb})$.

years. These have applications in laboratory plasmas, ionospheric experiments and space plasmas.

It may also be possible that an electrostatic pump wave may be upconverted into an electromagnetic mode via a similar mechanism, which should be very relevant to microwave emissions from space plasmas. However, this is only one of the many non-linear processes relevant for whistler waves. The growth rate of the parametric instability peaks around $\omega \cong \omega_\mathrm{cb}$. It requires a threshold pump power such that $u^2/c_\mathrm{s}^2 = 0.16$ where $u = eE_{0x}/m\omega_\mathrm{c}$, $c_\mathrm{s} = (T_\mathrm{e}/m_\mathrm{i})^{1/2}$.

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