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A review of non-commutative gauge theories

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Abstract. Construction of quantum field theory based on operators that are functions of non-commutative space-time operators is reviewed. Examples of ϕ^4 theory and QED are then discussed. Problems of extending the theories to $SU(N)$ gauge theories and arbitrary charges in QED are considered. Construction of standard model on non-commutative space is then briefly discussed. The phenomenological implications are then considered. Limits on non-commutativity from atomic physics as well as accelerator experiments are presented.

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1. Non-commutative space-time

The idea that coordinates may not commute can be traced back to Heisenberg. The earliest published reference is the work of Snyder [1] who acknowledges Heisenberg's role. The mathematical development of non-commutative geometry also has a long history [2]. Here we will review how quantum field theory is constructed such that it is consistent with non-commuting space-time operators. The fundamental postulate is

$$\left[\hat{X}_\mu \hat{X}_\nu\right] = i\theta_{\mu\nu} \tag{1}$$

where $\theta_{\mu\nu}$ is anti-symmetric c-number. The indices μ, ν run over time and space, although one could consider special cause where $\theta_{0i} = 0$. Non-commuting time operator might lead to violation of unitarity. The basic problem we want to consider is how to construct a quantum field theory which in the limit of $\theta_{\mu\nu} \to 0$ reverts to the standard model.

It should be clear from eq. (1) that $\theta_{\mu\nu}$ has the dimension of M^{-2}, and that the theory necessarily violates Lorentz invariance. Thus the effects of violation of Lorentz invariance are associated with a (large) scale M. There are two three-vectors associated with $\theta_{\mu\nu}$, $\theta_{0i} = A_i$ and $\varepsilon_{ijk}\theta_{ij} = B_k$. These two vectors will point along special directors with respect to fixed stars in our universe.

1.1 *An example from quantum mechanics: Landau problem*

Consider a particle of m and unit charge moving in an external uniform magnetic field in the direction z. If the motion in the x–y plane is considered, the Lagrangian is given by [3]

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \vec{v}\cdot\vec{A} \tag{2}$$

where

$$\vec{A} = -\frac{B}{2}(y, -x) \tag{3}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = (0, 0, B). \tag{4}$$

Lagrangian is

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{B}{2}\left(\dot{x}y - \dot{y}x\right). \tag{5}$$

In the limit of large B field or equivalently $m \to 0$, we have

$$L = \frac{B}{2}\left(\dot{x}y - \dot{y}x\right) \tag{6}$$

leading to x and y as canonically conjugate variables

$$[x, y] = \frac{i2}{B}. \tag{7}$$

The limit $m \to 0$ corresponds to the projection of the quantum mechanical spectrum to the lowest Landau level.

1.2 *String theory origin of non-commutation*

If one writes action of Neveu–Schwartz open string moving in a flat Eucliden space with metric g_{ij} in the presence of a constant background field B, we have

$$S_\Sigma = \frac{1}{2\pi\alpha'}\int_\Sigma \left(g_{ij}\partial_a x^i \partial_a x^j - 2\pi i\alpha' B_{ij}E^{ab}\partial_a x^i \partial_b x^j\right). \tag{8}$$

Here Σ is the string world sheet. The second term can be written as an integral over the boundary J the world sheet

$$= -\frac{i}{2}\oint B_{ij}x^i\partial_t x^j. \tag{9}$$

In the limit $g_{ij} \sim (\alpha')^2 \to 0$ with B fixed, only the second term survives. This is the low energy limit of string theory. Canonical quantization then gives non-commutative relation

$$[x_i, x_j] = (i/B)_{ij}. \tag{10}$$

In the limit of constant B field we get eq. (1). For a fuller discussion see [4].

2. Weyl quantization, star products and Moyal brackets

We review Weyl's method of introducing quantum operator associated with a classical function. This section is based on a recent review by Szabo [5].

A classical function, $f(x)$, assumed to be well behaved and real valued, has a Fourier transform given by

$$\tilde{f}(k) = \int \mathrm{d}^D x \mathrm{e}^{-ikx} f(x). \tag{11}$$

Then the operator field corresponding to $f(x)$ is defined by

$$\hat{f}(\hat{x}) = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \tilde{f}(k) \mathrm{e}^{ik_i\hat{x}^i}. \tag{12}$$

If we define $\hat{\Delta}(x)$ by

$$\hat{\Delta}(x) = \int \mathrm{e}^{-ik_i x^i} \mathrm{e}^{+ik_i\hat{x}^i} \frac{\mathrm{d}^D k}{(2\pi)^D} \tag{13}$$

then

$$\hat{f}(\hat{x}) = \int \mathrm{d}^D x\, f(x)\, \hat{\Delta}(x). \tag{14}$$

Further, using translational invariance,

$$\mathrm{Tr}\hat{\Delta}(x) = \mathrm{Tr}\hat{\Delta}(0) = 1 \quad \text{(normalized)}. \tag{15}$$

Thus

$$\mathrm{Tr}[\hat{f}(x)] = \int \mathrm{d}^D x\, f(x). \tag{16}$$

For products of two operator fields, we have

$$\hat{f}(\hat{x})\hat{g}(\hat{x}) = \int \mathrm{d}^D x\, f(x)\, \hat{\Delta}(x) \int \mathrm{d}^D y\, g(y)\, \hat{\Delta}(y). \tag{17}$$

In considering the product $\hat{\Delta}(x)\hat{\Delta}(y)$ we encounter products like $\mathrm{e}^{+ik_i\hat{x}^i}\mathrm{e}^{+ik'_i\hat{x}^i}$. Using Baker–Cambell–Hausdorff identity, and using $[\hat{X}_i\hat{X}_j] = i\theta_{ij}$, we have

$$\mathrm{e}^{ik_i\hat{x}^i}\mathrm{e}^{ik'_j\hat{x}^j} = \mathrm{e}^{-i/2\theta^{by}k_i k'_j}\mathrm{e}^{i(k+k')_i\hat{h}^i}. \tag{18}$$

We can easily establish the identity

$$\hat{f}(\hat{x})\hat{g}(\hat{x}) = \int \mathrm{d}^D x \hat{\Delta}(x)(f * g) \tag{19}$$

where star product is defined by

$$f(x) * g(x) = f(x) \exp\left(\frac{1}{2} \overleftarrow{\partial_i}\, \theta^{ij}\, \overrightarrow{\partial_j}\right) g(x). \tag{20}$$

Note that

$$\mathrm{Tr}\left[\hat{f}(\hat{x})\hat{g}(\hat{x})\right] = \int \mathrm{d}^D x (f * g). \tag{21}$$

For functions that vanish at $|x| \to \infty$, we have only in the case of bilinear traces by integrating by parts:

$$\int \mathrm{d}^D x (f * g) = \int \mathrm{d}^D x f g. \tag{22}$$

For products of three or more fields, we have the cyclic property which follows from (21)

$$\int \mathrm{d}^D x (f * g) * h = \int \mathrm{d}^D x h * (f * g). \tag{23}$$

Also, the star product shown to be associative, is

$$(f * g) * h = f * (g * h). \tag{24}$$

Moyal bracket is defined by

$$[f, g]_* = f * g - g * f = 2i f(x) \sin\left(\frac{1}{2} \overleftarrow{\partial_i}\, \theta^{ij}\, \overrightarrow{\partial_j}\right) g(x). \tag{25}$$

It is also useful to define 'derivative'

$$\left[\hat{\partial}_i, \hat{X}_j\right] = \delta_{ij} \tag{26}$$

$$\left[\hat{\partial}_i, \hat{\partial}_j\right] = 0. \tag{27}$$

Then we can easily show

$$\left[\hat{\partial}_i, \hat{\Delta}(x)\right] = \partial_i \hat{\Delta}(x) \tag{28}$$

and

$$\left[\hat{\partial}_i, \hat{f}\right] = \int \mathrm{d}^D x \left(\partial_i f_{(x)}\right) \hat{\Delta}(x). \tag{29}$$

Reader can refer to [5] for other useful properties.

3. Field theory examples

3.1 ϕ^4 *Theory*

The action can be written as

$$S = \mathrm{Tr}\left(\frac{1}{2}\left[\hat{\partial}_\mu, \hat{\phi}\right]^2 + \frac{m}{2}\hat{\phi}^2 + \frac{g^2}{4!}\hat{\phi}^4\right). \tag{30}$$

After carrying out the trace, we get

$$S = \int \mathrm{d}^D x \left[\frac{1}{2}(\partial_i, \phi)^2 + \frac{m}{2}\phi^2(x) + \frac{g^2}{4!}\phi(x) * \phi(x) * \phi(x) * \phi(x)\right]. \tag{31}$$

The interactions have higher derivative terms arising from the star product. The one-loop self energy of this ϕ^4 theory can be easily computed. There are two contributions at one loop, one the usual ϕ^4 contributions and the other dependent on θ_{ij}.

$$\pi_1(p) = \frac{1}{3}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2 + m^2} \tag{32}$$

$$\pi_2(p) = \frac{1}{6}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\mathrm{e}^{ik_i\theta^{ij}p_j}}{k^2 + m^2}. \tag{33}$$

First is ultraviolet divergent, the second converges for $p \neq 0$, but as $p \to 0$, it too diverges. This is referred to as infrared problem in non-commutative theories. We see that the theory is no better behaved than the usual ϕ^4 theory.

3.2 *Quantum electrodynamics*

Following Hayakawa [6], we define gauge transformation of a vector field as follows:

$$A_\mu(x) \to A'_\mu(x) = U(x) * A_\mu(x) * U^{-1}(x) + iU(x) * \partial_\mu U^{-1}(x) \tag{34}$$

where $U^{-1}(x)$ is defined with respect to star product by

$$U(x) * U^{-1}(x) = 1. \tag{35}$$

We define field strength $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu * A_\nu - A_\nu * A_\mu\right]. \tag{36}$$

Using the identity

$$\partial_\mu U(x) * U^{-1}(x) + U(x) * \partial_\mu U^{-1}(x) = 0 \tag{37}$$

we can show after some algebra that under gauge transformation $F_{\mu\nu}$ transforms as

$$F_{\mu\nu} \to U * F_{\mu\nu} * U^{-1}. \tag{38}$$

Gauge invariant action can be defined as

$$S = -\frac{1}{g^2}\int \mathrm{d}^D x F_{\mu\nu} * F^{\mu\nu}. \tag{39}$$

Note that unlike the usual QED, $F_{\mu\nu}$ is not gauge invariant although the action is, and $F_{\mu\nu}$ has photon self coupling like Yang–Mills theory.

Explicit realization of U and U^{-1} are

$$U(x) = \mathrm{e}^{i\alpha(x)}\Big|_* = 1 + i\alpha + \frac{i^2}{2!}\alpha * \alpha + \cdots \tag{40}$$

$$U^{-1}(x)\ e^{-i\alpha(x)}\Big|_* = 1 - i\alpha + \frac{(-i)^2}{2!}\alpha * \alpha + \cdots \tag{41}$$

Fermions interacting with photons can be introduced through the action

$$S_{\mathrm{F}} = \int \mathrm{d}^D x \left[i\bar{\psi}\gamma^\mu * D_\mu \psi - m\bar{\psi} * \psi\right] \tag{42}$$

where

$$D_\mu \psi = \partial_\mu \psi + iA_\mu * \psi. \tag{43}$$

Action is invariant under the transformation

$$\psi \to U * \psi \tag{44}$$

$$\bar{\psi} \to \bar{\psi} * U^{-1} \tag{45}$$

$$A_\mu \to U * A_\mu * U^{-1} + iU * A_\mu U^{-1}. \tag{46}$$

Note that there is no freedom to choose arbitrary charges for the fermion (except for the sign). The charge gets quantized. Gauge invariance requires that three photon coupling g be the same as fermion coupling to photon. This is a source of problem in generalizing this theory to the standard model where quarks have fractional charges. The Feynman rules for this type of QED are given in [7] and phenomenological application have been considered.

Another problem one encounters in generalizing this theory to non-ablian groups is that theory works for $U(N)$ groups but not for $SU(N)$.

To see the source of this problem, we can generalize the fields to matrix valued fields as follows:

$$A_\mu = A_\mu^a T^a \tag{47}$$

where T^a are representation of a group $SU(N)$.

Gauge transformation can be generalized as follows:

$$A_\mu \to U * A_\mu * U^{-1} + iU * \partial_\mu U^{-1} \tag{48}$$

where now

$$U = \exp\left(i\alpha^a T^a\right)\big|_* \tag{49}$$

$$U^{-1} = \exp(-i\alpha^a T^a)\big|_* . \tag{50}$$

Under an infinitesimal transformation the field transforms as

$$\delta\left[A_\mu\right] = \partial_\mu \alpha + i\left[\alpha, A_\mu\right]_* \tag{51}$$

expanding the Moyal bracket, we have two types of terms

$$\left[T^a, T^b\right] = f^{abc} T^C \tag{52}$$

which are within the groups $SU(N)$, and

$$\left\{T^a, T^b\right\} \tag{53}$$

which are in $U(N)$ but not $SU(N)$. We shall briefly present how these difficulties can be overcome.

3.3 *Non-commuting SU(N) gauge theory*

The difficulty of constructing $SU(N)$ theory is that the infinitesimal transformation seems to imply that δA lives in the enveloping algebra of the $SU(N)$ group. Jurčo *et al* [8] show how to define gauge transformations in this larger algebra that depend only on the gauge parameters of the group $SU(N)$.

If we define infinitesimal transformations as

$$\delta_\alpha \psi = i\Lambda_\alpha * \psi \tag{54}$$

$$\delta_\alpha A_\mu = \partial_\mu \Lambda_\alpha + i\left[\Lambda_\alpha, A_\mu\right]_* . \tag{55}$$

Then we must impose consistency condition on Λ such that

$$\left(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha\right)\psi = \alpha_a \beta_b f^{abc} T^C \psi. \tag{56}$$

By making a power series expansion, one can show that this constraint can be satisfied if

$$\Lambda_\alpha = \alpha + \frac{1}{4}\theta^{\mu\nu}\left\{\partial_\mu \alpha, A_\nu^0\right\} + 0\left(\theta^2\right) \tag{57}$$

where $A_\nu^0 \equiv A_{\nu^a}^0 T^a$ is the conventional $SU(N)$ gauge field. Similarly the fields ψ and A_μ have an expansion

$$A^{\mu} = A^{0\mu} - \frac{1}{4}\theta_{\rho\nu}\left\{A^{0\rho}, \partial^{\nu} A^{0\mu} + F^{0\nu\mu}\right\} \tag{58}$$

$$\psi = \psi^{0} - \frac{1}{2}\theta^{\mu\nu} A^{0}_{\mu}\partial_{\nu}\psi^{0} + \frac{i}{4}\theta^{\mu\nu} A^{0}_{\mu} A^{0}_{\nu}\psi^{0}. \tag{59}$$

By substituting for ψ and A in the action

$$S = \int \mathrm{d}^4 x \left[\bar{\psi} * (i\gamma_{\mu} D^{\mu} - m)\psi - \frac{1}{2g^2}\mathrm{Tr} F_{\mu\nu} * F^{\mu\nu}\right] \tag{60}$$

one gets action in terms of commuting fields ψ^0 and A^0_{μ}. This is the Seiberg–Witten map for non-commuting $SU(N)$. QED with fermions of arbitrary charges can be constructed following [9]

$$S = \sum_{N} \frac{1}{N} \int \mathrm{d}^D x f^{(n)}_{\mu\nu} * f^{(n)\mu\nu} + \int \bar{\psi}^{n} * \left(i\gamma^{\mu} D_{\mu} - m\right)\psi^{n} \tag{61}$$

where $\psi^{(n)}$ are fermions of charge $q^{(n)}$ and the physical electromagnetic field A_{μ} given by

$$A^{(n)}_{\mu} = A_{\mu} + \frac{eq^{(n)}}{4}\theta^{\alpha\beta}\left\{A_{\beta}, \partial_{\alpha} A_{\mu}\right\} + \frac{eq^{(n)}}{4}\theta^{\alpha\beta}\left\{f_{\alpha\mu}, A_{\nu}\right\} + 0\left(\theta^2\right). \tag{62}$$

3.4 *Bounds on θ from atomic spectroscopy*

At one loop, quarks interacting with gluons in a non-commutative theory lead to effective interactions of the form $\theta^{\mu\nu}\bar{q}\sigma_{\mu\nu}q$. This leads to an effective coupling of nuclear spin with the direction of space given by $\varepsilon_{ijk}\theta^{jk}$. If we assume the effective nuclear spin interaction is

$$\mathscr{L} = f\theta^{\mu\nu}\bar{N}\sigma_{\mu\nu}N \tag{63}$$

where f has dimension of M^3, we can derive the most stringent bound from the experiment of Berglund *et al* [10]. This experiment compares magnetic field measured by Cs and Hg atoms. Hg is sensitive to nuclear spin coupled to θ, while in Cs, the spin is due to electron and effect is much smaller. As the earth rotates, there will be a sidereal variation in the difference of the magnetic field measured by Cs and Hg. Absence of such a variation has been established at 110 nHz level. This implies

$$f|\theta| < 2\pi\hbar(110\ \mathrm{nHz}) \tag{64}$$

or

$$f|\theta| < 4.5 \times 10^{-31}\ \mathrm{GeV}. \tag{65}$$

Mocioiu *et al* [11] used $f \sim 0.05\ \mathrm{GeV}^3$ to derive a bound $|\theta| > 3 \times 10^{14}$ GeV. However it is difficult to estimate f since the one-loop integral in calculating θ coupling to quarks actually diverges quadratically. Calculation in non-commutative QCD [12] gives

$$f = \frac{\alpha_s}{12\pi}\Lambda^2 m \tag{66}$$

where $m \approx 300$ MeV in the constituent quark mass. This gives

$$|\theta|\Lambda^2 \lesssim 10^{-29}. \tag{67}$$

If the ultraviolet cut off Λ is taken as 1 TeV where new physics may set in, then one finds a very unnatural value of $\theta^{-1/2} \approx 10^{17}$ GeV. Clearly one should have liked $|\theta|\Lambda^2$ of order 1.

3.5 *Non-commutative standard model*

The standard model based on the gauge group $SU(3)\times SU(2)\times U(1)$ has been constructed by Calmet *et al* [9]. It is found that the presence of $U(1)$ causes ambiguities. The complete theory is characterized by six couplings, which reduce to the standard g_1, g_2 and g_3 when $\theta \to 0$. The additional couplings are responsible for Lorentz violating process such as $Z \to \gamma\gamma, Z \to gg$ and triple photon coupling [13]. There is a choice of couplings for which all these triple couplings are zero. Further, when the standard model is imbedded in a grand unified theory, depending on the GUT structure, all the couplings are uniquely fixed [14].

4. Conclusions

Non-commutative field theories provide an interesting study of theories that violate Lorentz symmetry. Origin of this violation is in the background field in the string theories. Both precise atomic experiments and forbidden processes provide proves of these theories. Accelerator experiments probe scales of only a few TeV while atomic experiments probe scales of 10^{14} GeV. If one assumes that θ are slowly varying fields, then accelerator experiments provide useful bounds as the atomic experiments assume constancy of θ over sidereal times.

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