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Quantum Einstein's equations and constraints algebra

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Abstract. In this paper we shall address this problem: Is quantum gravity constraints algebra closed and what are the quantum Einstein's equations. We shall investigate this problem in the de-Broglie–Bohm quantum theory framework. It is shown that the constraint algebra is weakly closed and the quantum Einstein's equations are derived.

Keywords. Quantum gravity; casual quantum theory; constraints algebra.

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1. Introduction

de-Broglie–Bohm causal quantum mechanics [1] has several positive points: (a) First of all, it is causal and so describes the system in an ordered way in time. (b) Perhaps the most important concept in the de-Broglie–Bohm theory is the lack of need for the assumption of the existence of a classical domain in the measurement phenomena. (c) This theory provides a useful framework for quantum gravity. (d) It does not suffer from the conceptual problems like the meaning of the wave function for a single system and so on. In fact there are a number of such positive points of this theory which can be found in the literature [2].

The application of this theory to quantum gravity has been investigated from different aspects: The quantum force may be repulsive and thus can remove the initial singularity and inflation would be emerged. Since the quantum domain is defined as the domain that the quantum force is smaller than the classical force, it is possible to have the classical universe for the small scale factors and conversely quantum universe for large scale factors [3]. As the scale factor represents the universe radius, there is not a one-to-one correspondence between large and classical universes. Because of the guidance formula, the time parameter appears automatically and the time problem does not exist. Moreover a new approach based on the de-Broglie–Bohm theory has been presented that brings out many interesting physical results, such as unification of quantal and gravitational behaviour of matter [4].

In the ADM decomposition of the space–time in general relativity the non-dynamical nature of shift and lapse functions (these are functions used in the slicing of the space–time and will be introduced later) must be consistent with the evolution. This is obviously satisfied if the constraint algebra is closed. Moreover, we know the secondary constraints in the ADM decomposition of the space–time are energy and momentum constraints. These are four non-dynamical Einstein's equations which together with the closedness of constraint algebra provides necessary and sufficient conditions for lapse and shift function to be non–dynamical as the space–time evolves. This is satisfied at the classical level as we know that the general relativity is independent of the space–time reparametrization. This property is guaranteed by the geometrical Bianchi identities.

In the present work we want to discuss this point at the quantum level. In order to do so, first we shall derive the constraint algebra at the quantum level. We shall use the integrated version of diffeomorphism and Hamiltonian constraints. In the other sections we shall derive the quantum Einstein's equations choosing arbitrary lapse and shift functions, i.e. in an arbitrary gauge. These equations are the extended form of those previously discussed by us [5].

2. Quantum constraints algebra

In the ADM formulation, the Hamiltonian of general relativity is

$$H = \int \mathrm{d}^3x \mathscr{H} \tag{1}$$

in which

$$\mathscr{H} = N\mathscr{H}_0 + N^i \mathscr{H}_i, \tag{2}$$

where

$$\mathscr{H}_0 = G_{ijkl}\Pi^{ij}\Pi^{kl} + \sqrt{h}(\mathscr{R} - 2\Lambda) \tag{3}$$

and

$$\mathscr{H}_i = -2\nabla_j \Pi^j_i. \tag{4}$$

In these equations, $\vec{N}$ is the shift function and N is the lapse function. These functions produce the time evolution of space-like surfaces in normal and tangent directions respectively. G_{ijkl} is the superspace metric and is given by

$$G_{ijkl} = \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right), \tag{5}$$

where h_{ij} is the three dimensional space-like hypersurface metric, Π^{ij} the conjugate canonical momentum of h_{ij}, $\mathscr{R}$ the intrinsic curvature of the hypersurface and Λ the cosmological constant. By Guass–Codazzi relations one can show that

$$\mathscr{H}_0 = 0 \tag{6}$$

and

$$\mathscr{H}_i = 0 \tag{7}$$

are in fact the four non-dynamical (constraints) Einstein's equations. So the Hamiltonian density vanishes by the Einstein's equations. The other dynamical Einstein's equations can be derived by differentiating the first equation with respect to h_{ij}.

Corresponding to the two directions of time evolution of hypersurface (normal and tangent), the Hamiltonian can be separated as

$$\mathscr{C}_c(N) = \int \mathrm{d}^3 x N \mathscr{H}_0 \tag{8}$$

and

$$\widetilde{\mathscr{C}}_c(\vec{N}) = \int \mathrm{d}^3 x N^i \mathscr{H}_i. \tag{9}$$

Because of the above explanations, $\widetilde{\mathscr{C}}_c(\vec{N})$ is called diffeomorphism constraint while $\mathscr{C}_c(N)$ is called Hamiltonian constraint. It is instructive to see the algebra of constraints. Using the notation of [6], one obtains

$$\left\{\widetilde{\mathscr{C}}_c(\vec{N}), \widetilde{\mathscr{C}}_c(\vec{N}')\right\} = \widetilde{\mathscr{C}}_c(N^i \vec{\nabla} N'_i - N'^i \vec{\nabla} N_i), \tag{10}$$

$$\left\{\mathscr{C}_c(N), \mathscr{C}_c(N')\right\} = \widetilde{\mathscr{C}}_c(N \vec{\nabla} N' - N' \vec{\nabla} N), \tag{11}$$

$$\left\{\widetilde{\mathscr{C}}_c(\vec{N}), \mathscr{C}_c(N)\right\} = \mathscr{C}_c(\vec{N} \cdot \vec{\nabla} N). \tag{12}$$

To quantize according to the standard quantum mechanics, one can use the Dirac quantization procedure which leads to

$$\widehat{\mathscr{C}(N)}\Psi = 0, \tag{13}$$

$$\widehat{\widetilde{\mathscr{C}}(\vec{N})}\Psi = 0. \tag{14}$$

These are quantum constraint and limit the physical wave function. The former is the WDW equation and the latter represents the invariance under general spatial transformation.

Now we shall apply the de-Broglie–Bohm theory to canonical quantum gravity. In the Hamilton–Jacobi language (which is suitable for our discussion), in de-Broglie–Bohm theory the desired quantum system is subjected to quantum potential in addition to the classical ones. This term includes all the quantum information about the system. It is a non-local potential and obtained by the norm of the wave function. By this simple description of this theory, we can discuss the constraints algebra at the quantum level.

As discussed in [2,5], the following change in $\mathscr{H}_0$ alone will be sufficient for description at quantum level.

$$\mathscr{H}_0 \to \mathscr{H}_0 + Q, \tag{15}$$

where Q is the quantum potential. The constraint equations corresponding to eqs (6) and (7) are

$$G_{ijkl}\Pi^{ij}\Pi^{kl} + \sqrt{h}(\mathscr{R} - 2\Lambda) + Q = 0, \tag{16}$$

$$-2\nabla_j \Pi_i^j = 0. \tag{17}$$

Dynamical equations are derived by differentiating the first equation and using Gauss–Codazzi relations, as this is done in the next section.

Equivalently we have

$$\mathscr{C}(N) = \mathscr{C}_c(N) + \mathscr{Q}(N), \tag{18}$$

$$\widetilde{\mathscr{C}}(\vec{N}) = \widetilde{\mathscr{C}}_c(\vec{N}), \tag{19}$$

where

$$\mathscr{Q}(N) = \int \mathrm{d}^3 x N Q = \int \mathrm{d}^3 x N \left(-\frac{\hbar^2}{\sqrt{h}|\Psi|} G_{ijkl} \frac{\delta^2 |\Psi|}{\delta h_{ij} \delta h_{kl}} \right). \tag{20}$$

Since at the quantum level $\widetilde{\mathscr{C}}(\vec{N})$ has not changed relation (10) is satisfied again. Some calculations leads to the following algebraic relations:

$$\left\{ \widetilde{\mathscr{C}}(\vec{N}), \widetilde{\mathscr{C}}(\vec{N}') \right\} = \widetilde{\mathscr{C}}(N^i \vec{\nabla} N'_i - N'^i \vec{\nabla} N_i), \tag{21}$$

$$\begin{aligned} &\left\{ \mathscr{C}(N), \mathscr{C}(N') \right\} = \widetilde{\mathscr{C}}_c(N \vec{\nabla} N' - N' \vec{\nabla} N) \\ &+2 \int \mathrm{d}^3 z \mathrm{d}^3 x \sqrt{h(z)} G_{ijkl}(z) \Pi^{kl}(z) \left(-N(z) N'(x) + N(x) N'(z) \right) \frac{\delta Q(x)}{\delta h_{ij}(z)} \approx 0, \end{aligned} \tag{22}$$

$$\left\{ \widetilde{\mathscr{C}}(\vec{N}), \mathscr{C}(N) \right\} = \mathscr{C}(\vec{N} \cdot \vec{\nabla} N). \tag{23}$$

The relation (23) is the same as the classical one and the relation (22) is weakly zero (≈ 0), i.e. zero only when the equation of motion is used. To see this, one must evaluate the derivative of quantum potential from the equation of motion (16) as

$$\frac{\delta Q(x)}{\delta h_{ij}(z)} = \frac{3}{4\sqrt{h}} h_{kl} \Pi^{ij} \Pi^{kl} \delta(x-z) - \frac{\sqrt{h}}{2} h^{ij} (\mathscr{R} - 2\Lambda) \delta(x-z) - \sqrt{h} \frac{\delta \mathscr{R}}{\delta h_{ij}}. \tag{24}$$

Using the known identity

$$F\frac{\delta\mathscr{R}(x)}{\delta h_{ij}(z)} = \left(-F\mathscr{R}^{ij} + \nabla^i\nabla^j F - h^{ij}\nabla^2 F\right)\delta(x-z), \tag{25}$$

where F is any arbitrary function, and substituting these relations in the Poisson bracket (22), shows that it is weakly zero. This means the one parameter family of diffeomorphism of spatial slices is a symmetry of the quantum space–time. But pushing spatial slices in the normal direction is a symmetry satisfied only at the level of quantum equations of motion. This point confirms our previous result ([5] and its references).

Therefore, as at the classical level, we must choose the initial conditions for space-like metric, lapse and shift functions to be consistent with the symmetry of space-time. Since the constraint algebra is not closed strongly, the Hamiltonian is not invariant under reparametrization of space–time. Then under dynamical evolution the symmetry does not survive. In this way different identical conditions lead to different solutions.

Because of the lack of invariance under time reparametrization we expect the general covariance symmetry to be broken, but this is not obvious at the level of the equations of motion. If we are interested in seeing the symmetry breaking at this level, we must look at quantum Einstein's equations. This point is explained in the next section.

3. Quantum Einstein's equations

Previously [5] in the Bohmian quantum gravity framework, we have studied the modifications of Einstein's equations in some special gauge. It was shown that the correction terms contain the quantum potential as we expected. Our discussion in [5] is based on ADM decomposition of the space–time and Gauss–Codazzi equations. But there it was assumed that the lapse function is 1 and the shift is zero for simplification. Here we derive the modified Einstein's equations in the general case.

The Gauss–Codazzi equations for any choice of lapse and shift functions are

$$G_{\mu\nu}n^\mu n^\nu = -\frac{1}{2}\left(\mathscr{R} + K^2 - K_{ij}K^{ij}\right) \tag{26}$$

and

$$G_{\mu i}n^\mu = \nabla_j K_i^j - \nabla_i K_i^j, \tag{27}$$

where

$$n_\mu = \frac{1}{N}(1, -\vec{N}) \tag{28}$$

represents a field of time-like vectors normal to a space-like slice. K_{ij} and $\mathscr{R}_{ij}$ are extrinsic curvature and Riemann tensor of three metric respectively. If we express the constraints quantities, $\mathscr{C}$ and $\widetilde{\mathscr{C}}$, in terms of the intrinsic curvature, using the following formula for conjugate momenta

$$\Pi^{ij} = \sqrt{h}(K^{ij} - h^{ij}K), \tag{29}$$

we find

$$\mathscr{H}_0 = -2G_{\mu\nu}n^\mu n^\nu, \tag{30}$$

$$\mathscr{H}_i = -2G_{\mu i}n^\mu. \tag{31}$$

Now according to de-Broglie–Bohm quantum theory of gravity we have

$$-2G_{\mu\nu}n^\mu n^\nu - Q = 0, \tag{32}$$

$$G_{\mu i}n^\mu = 0. \tag{33}$$

In continuation we use the form of general relativity action in terms of three metric and extrinsic curvature

$$\mathscr{A} = -\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}(\mathscr{R} + K^2 - K_{ij}K^{ij}). \tag{34}$$

Using the relations (34), (26), and (32) the dynamical equation of three metric is obtained as

$$\mathscr{G}^{ij} = \frac{\delta\mathscr{Q}}{\delta h_{ij}}. \tag{35}$$

As we expected, in the right hand side of the above dynamical equation the quantum force is appeared. From this relation by using eqs (32) and (33) we get the other remaining Einstein's equations

$$\mathscr{G}^{0i} = NN^iQ + N_j\frac{\delta\mathscr{Q}}{\delta h_{ij}}, \tag{36}$$

$$\mathscr{G}^{00} = N\left(N_iN^i - \frac{N}{2}\right)Q + N_iN_j\frac{\delta\mathscr{Q}}{\delta h_{ij}}. \tag{37}$$

It is simply seen that these equations are general form of the results of [5]. Thus in the modified Einstein's equations in a general case both shift and lapse functions appear. This motivates us to doubt about covariance under general coordinate transformations. A simple calculation shows that these equations are not covariant. Setting the right hand side of modified Einstein's equations as the components of a matrix $\mathscr{X}^{\mu\nu}$, to show the lack of general covariance one must investigate the transformation properties of $\mathscr{X}^{\mu\nu}$. Specifying the transformation to one in which $t \to t'(t)$ and $\vec{x}$ does not change, we have (using the transformation law of the metric)

$$h'_{ij} = h_{ij}, \tag{38}$$

$$N' = FN, \tag{39}$$

$$N'^i = FN^i \tag{40}$$

with

$$F = \frac{\partial t}{\partial t'} \tag{41}$$

and the quantum potential would remain unchanged. So we have

$$\mathscr{X}'^{00} = FN\left(F^2 N_i N^i - \frac{FN}{2}\right)Q + F^2 N_i N_j \frac{\delta \mathscr{Q}}{\delta h_{ij}}, \tag{42}$$

$$\mathscr{X}'^{0i} = F^2 N N^i Q + F N_j \frac{\delta \mathscr{Q}}{\delta h_{ij}}, \tag{43}$$

$$\mathscr{X}'^{ij} = \mathscr{X}^{ij}, \tag{44}$$

which shows that $\mathscr{X}^{\mu\nu}$ does not transform as a second rank tensor. Thus the general covariance principle is broken at the quantum level.

4. Conclusion

The quantum effects can be studied in gravity, as well as any other theory, by introducing the quantum potential. Since the quantum potential modifies the Hamiltonian, there is no guarantee that the constraints algebra be closed as it is in the classical case. The constraints algebra is in fact closed only weakly, i.e. by using the equations of motion. This shows that the associated symmetry should break down. We saw how one can obtain the equations of motion, i.e. the quantum Einstein's equations and how they are not general covariant.

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