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Gravitational field of spherical domain wall in higher dimension

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Abstract. An exact solution of Einstein's equations is found describing the gravitational field of a spherical domain wall with nonvanishing stress component in the direction perpendicular to the plane of the wall. Also we have studied the motion of test particle around the domain wall.

Keywords. Spherical domain wall; higher dimension; test particle.

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1. Introduction

The idea of higher dimensional theory was originated in super string and super gravity theories to unify gravity with other fundamental forces in nature. Developments in super string theories have stimulated the study of physics in higher dimensional space times [1]. Solutions of Einstein's field equations in higher dimensional space times are believed to be of physical relevance possibly at the extremely early times before the universe underwent the compactification transitions. At the early stages of its evolution, the universe undergoes phase transitions and as a result several topological defects appeared [2]. Topological objects such as monopoles, strings and domain walls have an important role in the formation of our universe. In general relativity, domain walls (which are formed when the universe undergoes a series of phase transitions with a discrete symmetry being spontaneously broken) are getting special attention due to their peculiar and interesting gravitational effects. At first, Chodos *et al* [3] studied the cosmological implications of the higher dimensions by constructing a Kasner type vacuum space time in five dimension. So many works were done on domain walls [4]. Recently, Banerji *et al* and Patel *et al* [5] have studied thick domain walls in five dimensional space time. The purpose of this work is to study the gravitational field of spherically symmetric domain walls in five dimensional space time in the frame work of general relativity. Here the domain wall is characterized by the energy momentum tensor [5]

$$T_{\mu r} = \rho(g_{\mu r} + \xi_\mu \xi_r) + p_1 \xi_\mu \xi_r,$$

where p_1 is the pressure in the direction normal to the plane of the wall and ξ_μ is a unit space-like vector in the same direction. Also we study the motion of the test particle in the gravitational field of spherical domain wall using Hamilton–Jacobi formalism [6] and examine whether bound orbits are possible or not. This study will be of relevance to the structure formation because it gives some idea about the behaviour of the particles (created at the early universe) in the gravitational field of the domain walls. Our paper is organized as follows: The basic equations are constructed in §2 while the solutions to the field equations are presented in §3. The geodesic equations will be discussed in §4. The paper ends with a short discussion in §5.

2. Basic equations

To search for spherical domain wall in five dimensional space time we shall start with the general spherical symmetric metric as

$$\mathrm{d}s^2 = -A\mathrm{d}t^2 + B\mathrm{d}r^2 + C(d\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + E\,\mathrm{d}\psi^2, \tag{1}$$

where $A,\ B,\ C,\ E$ are functions of r and t and ψ are the fifth co-ordinate.

The energy stress components in comoving co-ordinates for the domain wall under consideration here are given by

$$T_r^t = 0, \quad T_t^t = T_\theta^\theta = T_\phi^\phi = T_\psi^\psi = \rho \quad \text{and} \quad T_r^r = -p_1, \tag{2}$$

where ρ is the energy density of the wall, which is again equal to the tension along θ,ϕ and ψ directions in the plane of the wall; p_1 is the pressure along the r direction.

Now Einstein's equations are

$$\begin{aligned}&\frac{1}{2B}\left(-\frac{2C''}{C} - \frac{E''}{E} + \frac{C'^2}{2C^2} + \frac{E'^2}{E^2} + \frac{B'C'}{BC} + \frac{E'B'}{2EB} - \frac{E'C'}{CE}\right)\\ &\quad + \frac{1}{2A}\left(\frac{\dot{C}^2}{2C^2} + \frac{\dot{E}\dot{C}}{EC} + \frac{\dot{B}\dot{C}}{BC} + \frac{1}{2}\frac{\dot{B}\dot{E}}{BE}\right) + \frac{1}{C} = 8\pi\rho,\end{aligned} \tag{3}$$

$$\begin{aligned}&\frac{1}{2B}\left(-\frac{C'^2}{2C^2} - \frac{A'E'}{2AE} - \frac{A'C'}{AC} - \frac{C'E'}{CE}\right)\\ &\quad + \frac{1}{2A}\left(\frac{2\ddot{C}}{C} + \frac{\ddot{E}}{E} - \frac{\dot{C}^2}{2C^2} - \frac{\dot{E}^2}{E^2} - \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{A}\dot{E}}{2EA} + \frac{\dot{E}\dot{C}}{EC}\right) + \frac{1}{C} = -8\pi p_1,\end{aligned} \tag{4}$$

$$\begin{aligned}&\frac{1}{2B}\left(-\frac{A''}{A} - \frac{C''}{C} - \frac{E''}{E} + \frac{A'^2}{2A^2} + \frac{C'^2}{2C^2} + \frac{E'^2}{2E^2} + \frac{A'B'}{2AB} + \frac{1}{2}\frac{B'C'}{BC} + \frac{E'B'}{2EB}\right.\\ &\quad \left. - \frac{A'C'}{2AC} - \frac{A'E'}{2AE} - \frac{C'E'}{2CE}\right) + \frac{1}{2A}\left(\frac{\ddot{C}}{C} + \frac{\ddot{E}}{E} - \frac{\dot{B}^2}{2B^2} - \frac{\dot{C}^2}{2C^2} - \frac{\dot{E}^2}{2E^2}\right.\\ &\quad \left. + \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{2AB} - \frac{\dot{A}\dot{C}}{2AC} - \frac{\dot{A}\dot{E}}{2AE} + \frac{\dot{B}\dot{E}}{2BE} + \frac{\dot{E}\dot{C}}{2EC} + \frac{\dot{B}\dot{C}}{2BC}\right) = 8\pi\rho,\end{aligned} \tag{5}$$

$$\frac{1}{2B}\left(-\frac{A''}{A}-\frac{2C''}{C}+\frac{A'^2}{2A^2}+\frac{C'^2}{2C^2}+\frac{A'B'}{2AB}+\frac{B'C'}{BC}-\frac{A'C'}{AC}\right)$$
$$+\frac{1}{2A}\left(\frac{\ddot{B}}{B}+\frac{2\ddot{C}}{C}-\frac{\dot{C}^2}{2C^2}-\frac{\dot{A}\dot{B}}{2AB}-\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}-\frac{\dot{B}^2}{2B^2}\right)+\frac{1}{C}=8\pi\rho, \tag{6}$$

$$\frac{1}{B}\left(\frac{\dot{C}'}{C}+\frac{\dot{E}'}{2E}-\frac{1}{2}\frac{\dot{C}C'}{C^2}-\frac{\dot{E}E'}{4E^2}-\frac{A'\dot{C}}{2AC}-\frac{A'\dot{E}}{4AE}-\frac{\dot{B}C'}{2BC}-\frac{\dot{B}E'}{4BE}\right)=0. \tag{7}$$

Here a dot and a prime denote partial differentiation with respect to t and r respectively.

The general solutions for this space time are apparently quite difficult to obtain.

We shall now solve the field equations assuming separable form of the metric coefficients as follows:

$$A=A_1(r)A_2(t);\quad B=B_1(r)B_2(t);\quad C=C_1(r)C_2(t);\quad E=E_1(r)E_2(t). \tag{8}$$

3. Solutions to the field equations

From (7), using the separable form (8), we get

$$\frac{\dot{C}_2}{2C_2}\left(\frac{C_1'}{C_1}-\frac{A_1'}{A_1}\right)+\frac{\dot{E}_2}{4E_2}\left(\frac{E_1'}{E_1}-\frac{A_1'}{A_1}\right)-\frac{\dot{B}_2}{4B_2}\left(\frac{E_1'}{E_1}+\frac{2C_1'}{C_1}\right)=0. \tag{9}$$

Equation (9) leads to a possible choice

$$\frac{C_1'}{C_1}-\frac{A_1'}{A_1}=0 \tag{10}$$

so that

$$\frac{\dot{E}_2}{E_2}\bigg/\frac{\dot{B}_2}{B_2}=\frac{(E_1'/E_1)+(2C_1'/C_1)}{(E_1'/E_1)-(A_1'/A_1)}=m\ \text{(separation constant)}. \tag{11}$$

Thus

$$E_2=B_2^m \tag{12}$$

and

$$E_1=C_1^{m^1}, \tag{13}$$

where $m^1=(2+m)/(m-1)$.

Now, eliminating ρ from (3) and (6), we get

$$\frac{1}{2B}\left(-\frac{E''}{E}+\frac{E'^2}{E^2}+\frac{E'B'}{2EB}-\frac{E'C'}{CE}+\frac{A''}{A}-\frac{A'^2}{2A^2}-\frac{A'B'}{2AB}+\frac{A'C'}{AC}\right)$$
$$+\frac{1}{2A}\left(\frac{\dot{C}^2}{2C^2}+\frac{\dot{E}\dot{C}}{EC}+\frac{1}{2}\cdot\frac{\dot{B}\dot{E}}{BE}-\frac{\ddot{B}}{B}-\frac{2\ddot{C}}{C}+\frac{\dot{C}^2}{2C^2}+\frac{\dot{A}\dot{B}}{2AB}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}^2}{2B^2}\right)=0. \tag{14}$$

It is to be noted that without any loss of generality, one can take $B_1 = A_2 = 1$ (the values of B_1 and A_2 different from unity only results in a transformation of r and t co-ordinates). Putting the above results, we get from (14), the space part as

$$C_1'' - U\frac{C_1'^2}{C_1} = p', \quad \text{where} \quad U = \frac{1}{2(m'-1)}; \quad p' = \frac{p}{m'-1}, \tag{15}$$

where p is the separation constant.

Now, after integrating, we get from (15)

$$C_1'^2 = \frac{2p^1}{1-2U}C_1 + D_1 C_1^{2U} \quad \text{for } U \neq 1/2 \tag{16}$$

$$= 2p^1 C_1 \ln C_1 + D_1 C_1 \quad \text{for } U = 1/2, \tag{17}$$

where D_1 is the integration constant. Hence the integral forms of C_1 are

$$\pm(r - r_0) = \int \frac{\mathrm{d}C_1}{\left[\frac{2p'}{1-2U}C_1 + D_1 C_1^{2U}\right]^{1/2}} \quad \text{for } U \neq 1/2 \tag{18}$$

$$= \int \frac{\mathrm{d}C_1}{[2p' C_1 \ln C_1 + D_1 C_1]^{1/2}} \quad \text{for } U = 1/2. \tag{19}$$

We see that the above integral for $U = 1/2$ is not exactly solvable. So we will consider the cases for $U \neq 1/2$. The following are the situations where one can get the exact analytic form for $C_1(r)$.

Case I: $D_1 = 0$. In this case we can easily solve the integral (18) to give

$$C_1(r) \propto r^2. \tag{20}$$

Case II: $p = 0$. Here $C_1(r)$ is of the form

$$C_1(r) \propto r^{1/(1-U)}. \tag{21}$$

Case III: $U = 1$. The solution of the integral (18) is

$$C_1(r) = \frac{(\mathrm{e}^{\sqrt{D_1}r} + 4p)^2}{2D_1 \mathrm{e}^{\sqrt{D_1}r}}. \tag{22}$$

Case IV: $U = 1/4$. In this case we can easily solve (18) whereas we do not get the explicit form of $C_1(r)$. Here we get the expression of $C_1(r)$ as

$$r = \frac{(2pC_1 + D_1\sqrt{C_1})^{1/2}}{p} - \frac{D_1}{2p}\left[\frac{1}{\sqrt{2p}}\ln(16p^2 C_1 + 8pD_1\sqrt{C_1})^{1/2} + 4p\sqrt{C_1} + D_1\right]. \tag{23}$$

For time part, we introduce an assumption

$$B_2 = C_2^n, \tag{24}$$

where n is an arbitrary constant.

Now from (14), we get

$$\ddot{C}_2 - q\frac{\dot{C}_2^2}{C_2} = -q' C_2^{-n+1}, \tag{25}$$

where

$$q = \frac{mn + (mn^2/2) + n + 1}{n+2}; \quad q' = \frac{p}{n+2}.$$

After integrating, we get from (25)

$$\dot{C}_2^2 = \frac{2q'}{2q+n-2} C_2^{2-n} + D_2 C_2^{2q} \quad \text{for } n \neq 2-2q \tag{26}$$

$$= 2q' C_2^{-2q} \cdot \ln C_2 + D_2 C_2^{2q} \quad \text{for } n = 2-2q, \tag{27}$$

where D_2 is the integration constant.

The integral forms of C_2 are

$$\pm(t - t_0) = \int \frac{\mathrm{d}C_2}{\left[\frac{2q'}{2q+n-2} C_2^{2-n} + D_2 C_2^{2q}\right]^{1/2}} \quad \text{for } n \neq 2-2q \tag{28}$$

$$= \int \frac{\mathrm{d}C_2}{[2q' C_2^{-2q} \ln C_2 + D_2 C_2^{2q}]^{1/2}} \quad \text{for } n = 2-2q. \tag{29}$$

Here we also see that the above integral for $n = 2-2q$ is not solved exactly. So we will consider the cases for $n \neq 2-2q$.

Case I: $p = 0$. Here C_2 is of the form

$$C_2(t) \propto t^{1/(1-q)} \quad (q \neq 1). \tag{30}$$

Case II: $D_2 = 0$. In this case we get

$$C_2(t) \propto t^{2/n}. \tag{31}$$

Case III: $n = 0$. The solution of the integral (28) is

$$C_2(t) = \frac{1}{2p}\left(\sin\frac{\sqrt{p}}{2}t + \cos\frac{\sqrt{p}}{2}t\right)^2. \tag{32}$$

Case IV: $n = 2$, $q = 1$. Here C_2 has the expression as

$$C_2(t) = \frac{1}{4D_2}(D_2^2 t^2 - 2p). \tag{33}$$

Case V: $q = 1$, $n = 1$. We solve eq. (28) to get the expression of C_2 as

$$C_2(t) = \frac{(\mathrm{e}^{\sqrt{D_2}\,t} - 2q')^2}{2D_2 \mathrm{e}^{\sqrt{D_2}t}}. \tag{34}$$

Thus we have a family of solutions for different choices of arbitrary constants.

3.1 *Effective four dimensional theory*

It may be interesting to recast the whole formalism in an effective four dimensional background. One can note that the diagonal metric we obtained for a five dimensional space time corresponding to spherically symmetric domain wall appear as an effective four dimensional metric given by

$$\mathrm{d}s^2 = -C_1\mathrm{d}t^2 + C_2^n\mathrm{d}r^2 + C_1C_2\mathrm{d}\Omega_2^2. \tag{35}$$

We also note that this solution is exactly same as the one obtained by Farook *et al* [4] for choosing the suitable arbitrary constants in four dimensional case.

3.2 *Deficit angle*

For the case $n = 1$, we note that with a proper choice of radial and time co-ordinates, the domain wall metric can be written as (except for a conformal factor)

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}R^2 + \mathrm{d}\Omega_1^2 + C_1^{m^1-1}C_2^{m-1}\mathrm{d}\psi^2,$$

where $T = \int \mathrm{d}t/\sqrt{C_2}$ and $R = \int \mathrm{d}r^2/\sqrt{C_1}$.

So we get a solid angle of deficit which depends both on radial and time co-ordinates.

4. Motion of test particles

Let us consider a relativistic particle of mass m_0 moving in the field of the spherical domain wall metric (1) (for $n = 0$). So the Hamilton–Jacobi (H–J) equation is [6]

$$-\frac{1}{C_1(r)}\left(\frac{\partial s}{\partial t}\right)^2 + \left(\frac{\partial s}{\partial r}\right)^2 + \frac{1}{c_1(r)c_2(t)}\left[\left(\frac{\partial s}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial s}{\partial \phi}\right)^2\right] + \frac{1}{(C_1(r)^{m^1}}\left(\frac{\partial s}{\partial \psi}\right)^2 + m_0^2 = 0. \tag{36}$$

As the metric (1) is independent of ϕ and ψ co-ordinates, a natural choice for H–J function in separable form will be

$$s(r,\theta,\phi,\psi,t) = -S_1(t) + S_2(r) + S_3(\theta) + M\cdot\psi + J\phi, \tag{37}$$

where the constants J and M are identified as the angular momentum and five dimensional velocity respectively.

Now substituting the ansatz (37) in the H–J equation (36) we get the following expressions for unknown functions S_1, S_2 and S_3 as

$$S_1 = \varepsilon\int\left(\beta^2 + \frac{\alpha^2}{C_2}\right)^{1/2}\mathrm{d}t, \tag{38}$$

$$S_2 = \varepsilon \int \left(\frac{p^2}{C_1} - m_0 - \frac{M^2}{C_1^{m^1}} \right)^{1/2} \mathrm{d}r \tag{39}$$

and

$$S_3 = \varepsilon \int (\alpha^2 - J^2 \mathrm{cosec}^2\theta)^{1/2} \mathrm{d}\theta, \tag{40}$$

where α and β are separation constants and $\varepsilon = \pm 1$.

In the H–J formalism, the path of the particle is characterized by [6] $\partial s/\partial J =$ constant, $\partial s/\partial M =$ constant, $\partial s/\partial \beta =$ constant, $\partial s/\partial \alpha =$ constant.

Thus we get (taking the constants to be zero without any loss of generality),

$$\phi = \varepsilon \int J \,\mathrm{cosec}^2\theta (\alpha^2 - J^2 \,\mathrm{cosec}^2\theta)^{-1/2} \mathrm{d}\theta, \tag{41}$$

$$\psi = \varepsilon \int \frac{M}{C_1^{m^1}} \left(\frac{\beta^2}{C_1} - m_0^2 - \frac{M^2}{C_1^{m^1}} \right)^{-1/2} \mathrm{d}r, \tag{42}$$

$$\int (\alpha^2 - J^2 \mathrm{cosec}^2\theta)^{-1/2} \mathrm{d}\theta = \int \frac{1}{C_2} \left(\beta^2 + \frac{\alpha^2}{C_2} \right)^{-1/2} \mathrm{d}t, \tag{43}$$

$$\int \left(\beta^2 + \frac{\alpha^2}{C_2} \right)^{-1/2} \mathrm{d}t = \int \frac{1}{C_1} \left(\frac{\beta^2}{C_1} - m_0^2 - \frac{M^2}{C_1^{m^1}} \right)^{-1/2} \mathrm{d}r. \tag{44}$$

From (44), we get the radial velocity as

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{C_1[(\beta^2/C_1) - m_0^2 - (M^2/C_1^{m^1})]^{1/2}}{[\beta^2 + (\alpha^2/C_2)]^{1/2}}. \tag{45}$$

Now that the turning points of the trajectory are given by

$$\mathrm{d}r/\mathrm{d}t = 0, \tag{46}$$

we see that real solutions exist in this equation. So the orbit of a massive test particle is always bounded, i.e. particles can be trapped by spherical domain wall. Thus spherical domain wall exerts gravitational force which is attractive in nature.

5. Concluding remarks

Here we have studied the gravitational field of a spherical domain wall in a five-dimensional space time. We obtain a class of solutions which is exactly similar to that obtained by Farook *et al* [4] via a specific choice of the arbitrary constant in the ψ constant hypersurface. We can see that both energy density and pressure along r-direction depend on radial and time co-ordinates and both will vanish at $r \to \pm\infty$. An interesting result obtained from this work is that the pressure perpendicular to the wall is nonzero. The gravitational field of the spherical domain wall has been shown to be attractive by studying the motion of test particle in the gravitational field of the domain wall. The study of the dynamics of the spherical domain wall is the subject of future investigations.

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