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Parity violation in a gravitational theory with torsion: A geometrical interpretation

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Abstract. In a space-time with torsion, the action for the gravitational field can be extended with a parity-violating piece. We show how to obtain such a piece from geometry itself, by suitably modifying the affine connection so as to include a pseudo-tensorial part. A consistent method is thus suggested for incorporating parity-violation in the Lagrangians of all matter fields with spin in a space-time background with torsion.

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Theories of gravitation in a space-time with torsion have been under investigation for a long time. The presence of torsion modifies the spacetime geometry through non-symmetric connections [1]. Torsion has been incorporated in theories of gravitation, which range from ones as old as the Einstein–Cartan model to developments as recent as super-string theories.

Various consequences of torsion have already been explored in the literature [2], including those involving complex connections [3], and aiming, for example, to present the electromagnetic potential as the trace of the torsion tensor [4]. As a result of torsion, the standard Einstein–Hilbert action admits of an additional term which is *parity-violating* [5]. Thus a generalised theory of gravity can be conceived of as having the possibility of parity violation inbuilt in it. An immediate extension of this idea leads to the expectation that the Lagrangians of various types of matter fields coupled to a torsioned space-time background [6,7] will also contain parity-violating terms. Although such terms are *a priori* weaker than those corresponding to weak interacton (which is an already established source of parity violation), they can be significant in astrophysical and cosmological contexts [8]. In particular, as has been pointed out in some recent works [9], such parity-violation can reflect itself in observable quantities related to the cosmic microwave background radiation, such as polarization asymmetries and temperature maps. And, above all, the sheer universality of gravity compels us to attach sufficient importance to such new physics possibilities.

The question that perhaps may be raised is: is there a consistent way of introducing parity violation in a spacetime with torsion, which will be applicable to both pure gravity and the various types of matter fields with spin? Here we try to answer this question by seeking the origin of parity violation in the geometry of space-time [10]. We point out that just as the original Einstein–Cartan theory follows by adding an antisymmetric tensor to the hitherto symmetric affine connnections, the incorporation of a pseudo-tensorial connection introduces parity violation in the theory. This enables one to start from the original Einstein–Hilbert action, with the scalar curvature R suitably modified by the new form of the covariant derivative. We show that, with the most general choice of the pseudo-tensorial connection which is linear in the torsion field, the induced parity-violating term has close resemblance to what arises in earlier works from a separate piece in the Lagrangian. This generalised connection also enables us to obtain the Lagrangians for spin-1 and spin-1/2 fields, with built-in parity-violating interactions with the torsion field. Thus one can trace all parity-violating effects in torsioned space to a common origin.

The Einstein–Hilbert action for pure gravity is given by

$$S = \int \sqrt{-g} R \mathrm{d}^4 x \tag{1}$$

where R is the scalar curvature, defined as $R = R_{\alpha\mu\beta\nu} g^{\alpha\beta} g^{\mu\nu}$ and $R_{\alpha\mu\beta\nu}$ is the Riemann–Christoffel tensor.

In the absence of torsion, the Γ's are the usual Christoffel symbols, symmetric in the two lower indices.

If there is torsion, Γ needs to be replaced by $\tilde{\Gamma}$, where

$$\tilde{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} - H^{\mu}_{\nu\lambda} \tag{2}$$

H being antisymmetric in the two lower indices. Further, the requirement that the metric be covariantly conserved (the so-called 'metricity condition') restricts H to a form where it is antisymmetric in all three indices, a form in which it is commonly known as the 'contortion tensor'. It is related to the torsion field S through $H^{\lambda}_{\mu\nu} = -S^{\lambda}_{\mu\nu} + S_{\nu}{}^{\lambda}{}_{\mu} - S^{\lambda}_{\mu\nu}$.

The inclusion of H destroys the cyclicity of $R_{\alpha\beta\mu\nu}$ in any three of its four indices. Consequently, a term of the form $\epsilon^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu}$ (which, in a torsion-free scenario, would have been forced to vanish by the cyclicity condition) can be added to the scalar curvature [5], in perfect consistence with general covariance. The latter is manifestly parity-violating. Thus one is led to the conclusion that unless the conservation of parity is artificially imposed on the theory, there is no reason for it to hold generally in a scenario with torsioned space-time.

However, the above way of introducing the parity-violating terms under torsion is of a somewhat *ad hoc* nature. Nor does it provide one with a guideline as to how to extend this to the gravitational interactions of particles of different spins, once parity ceases to be a symmetry in the pure gravity sector.

Our proposal is to introduce parity violation in the covariant derivative in a curved space-time. That is to say, the most general connection can be made to include, in addition to the symmetric and antisymmetric parts, a pseudo-tensorial part as well. Remembering that the extra pieces should vanish in the limit of zero torsion, a general form in the minimal extension scheme is

$$\tilde{\Gamma}^{\kappa}_{\nu\lambda} = \Gamma^{\kappa}_{\nu\lambda} - H^{\kappa}_{\nu\lambda} - q(\epsilon^{\gamma\delta}_{\nu\lambda} H^{\kappa}_{\gamma\delta} - \epsilon^{\kappa\alpha}_{\beta\lambda} H^{\beta}_{\nu\alpha} + \epsilon^{\kappa\alpha}_{\beta\nu} H^{\beta}_{\lambda\alpha}). \tag{3}$$

Here q is a parameter determining the extent of parity violation, depending, presumably, on the matter distribution. It is extremely important to note at this juncture that the covariant derivative of the metric ($D_\mu g_{\nu\lambda} = 0$, with D_μ defined in terms of $\tilde{\Gamma}$) automatically vanishes so long as H is antisymmetric in all three indices. Therefore the same condition for metricity as the one in ordinary Einstein–Cartan theory also suffices when a pseudo-tensorial part is included in the connection. However, the relative signs of the three additional terms get fixed by metricity. The same requirement also leads us to the conclusion that all the three aforementioned terms have to be coupled through the same charge q.

Next, we calculate the curvature scalar $\tilde{R}$ using the above connection and the metricity condition. After some algebra, one obtains

$$\tilde{R} = R^{EC} + R^{X} \tag{4}$$

where R^{EC} is the ordinary Einstein–Cartan scalar curvature, which arises from the parity-conserving part of the covariant derivative. The artifacts of the pseudo-tensorial connection are found in R^X which is given by

$$R^X = R^{pv} + R^{pc} \tag{5}$$

with

$$R^{pv} = -6q\epsilon^{\sigma\nu}_{\alpha\beta} H^{\alpha\beta}_{\mu} H^{\mu}_{\sigma\nu} \tag{6}$$

and

$$R^{pc} = 3q^2[2\epsilon^{\gamma\delta}_{\lambda\sigma}\epsilon^{\sigma\eta}_{\omega\nu} H^{\nu}_{\gamma\delta} H^{\lambda\omega}_{\eta} + \epsilon^{\rho\eta}_{\nu\lambda}\epsilon^{\lambda\nu}_{\alpha\beta} H^{\alpha\beta}_{\sigma} H^{\sigma}_{\rho\eta}]. \tag{7}$$

The most striking conclusion from above is that the parity violating piece R^{pv} is, modulo the multiplicative factor $-6q$, *exactly same* as what one gets from the term $\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ added *ad hoc* to the Einstein–Hilbert–Cartan action. In addition, one would have in the latter case a term proportinal to the derivative of the H-field, which is absent in our case. It is obvious, however, that such a derivative term is bound to vanish whenever torsion can be expressed in terms of the derivative of any antisymmetric second-rank tensor field, *i.e.* $H_{\alpha\beta\gamma} = \partial_{[\alpha]} B_{[\beta\gamma]}$. This is precisely what one obtains in superstring theories. Therefore, the parity-violating terms obtained in variations of such theories following our approach match exactly with those arising from the term $\epsilon^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ as used in earlier works. Moreover, in our aproach, R^X contains terms quadratic in the Levi–Civita tensor density, which provide extra parity-conserving pieces over and above those present in the minimal Einstein–Cartan framework. These terms are, however, proportional to q^2, and therefore will be relatively unimportant when there is only a small amount of parity violation.

It is straightforward now to examine the spin-1 and spin-1/2 sectors. First, we take up the case of a spin-1 Abelian gauge field A. Here, one can write down a gauge invariant Lagrangian either in terms of a field tensor defined in the tangent space, or with additional non-linear terms introduced in the Lagrangian. We follow the latter approach, since it gives us a uniform procedure for all matter fields with spin. Hence we use the augmented connection, the corresponding field tensor being defined here as

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu \tag{8}$$

where the covariant derivative is to be written as $D_\mu A_\nu = \partial_\mu A_\nu + \tilde{\Gamma}^{\beta}_{\mu\nu} A_\beta$, which includes both the Cartan and pseudo-tensorial extensions. Thus the expression for $F_{\mu\nu}$ in our case turns out to be

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - 2H^{\beta}_{\mu\nu} A_\beta - 2q\epsilon^{\gamma\delta}_{\mu\nu} H^{\beta}_{\gamma\delta} A_\beta + 2q\epsilon^{\gamma\beta}_{\delta\nu} H^{\delta}_{\mu\beta} A_\gamma - 2q\epsilon^{\gamma\beta}_{\delta\mu} H^{\delta}_{\nu\beta} A_\gamma. \quad (9)$$

It is obvious that the Lagrangian for the gauge field, given by $-(1/4)F^{\mu\nu}F_{\mu\nu}$, now contains both terms linear and quadratic in ϵ. In addition, of course, this approach will require one to introduce addition interaction terms involving A_μ and $F^{\mu\nu}$ in order to preserve gauge invariance [7]. Extension of the formalism to a non-abelian gauge field is straightforward, with the possibility of additional parity violation in the self-coupling terms.

For a spin-1/2 field in a spacetime with torsion, again, the standard Dirac Lagrangian needs to be extended with the appropriate covariant derivative. It has the general form [6,11]

$$\mathcal{L}_{\rm Dirac} = \overline{\psi}[i\gamma^\mu(\partial_\mu - \sigma^{\rho\beta} v^\nu_\rho g_{\lambda\nu} \partial_\mu v^\lambda_\beta - g_{\alpha\delta}\sigma^{ab} v^\beta_a v^\delta_b \tilde{\Gamma}^\alpha_{\mu\beta})]\psi. \quad (10)$$

where one has introduced the tetrad v^μ_a to connect the curved space with the corresponding tangent space at any point. (The greek indices correspond to the curved space, and the Latin indices, to the tangent space.) Using the full form of $\tilde{\Gamma}$ defined in our scenario, $\mathcal{L}_{\rm Dirac}$ can be expressed as

$$\mathcal{L}_{\rm Dirac} = \mathcal{L}^E + \overline{\psi}[i\gamma^\mu g_{\alpha\delta}\sigma^{\beta\delta} H^\alpha_{\mu\beta}]\psi + \mathcal{L}^{pv}. \quad (11)$$

the first term being the same as what one would have gotten in Einstein gravity, and the second one corresponds to the Cartan extension. The incorporation of a pseudo-tensorial extension results in the parity-violating part $\mathcal{L}^{pv}$ which is given by

$$\mathcal{L}^{pv} = q(det g^{\eta\eta'})[-\overline{\psi}\gamma_\mu\sigma_{\lambda\omega}\gamma_5\psi H_{\gamma\rho\delta} + g_{\alpha\delta}\overline{\psi}\gamma^\mu\sigma_{\omega\nu}\gamma_5\psi H_{\mu\lambda\rho} - g_{\alpha\delta}\overline{\psi}\gamma_\mu\sigma_{\omega\nu}\gamma_5\psi H_{\beta\lambda\rho}]. \quad (12)$$

where $\eta(\eta')$ runs over $\{\delta, \lambda, \omega(\mu, \gamma, \rho)\}$, $\{\delta, \omega, \nu(\rho, \alpha, \lambda)\}$, and $\{\beta, \delta, \omega, \nu(\rho, \mu, \alpha, \lambda)\}$ in the first, second and third terms.

Note that here one does not get any additional pieces quadratic in the pseudo-tensorial connecton because of the very structure of the Dirac Lagrangian. Therefore, unlike in the case of pure gravity and the vector field, we do not end up with additional parity-conserving terms resulting from such a connection.

In conclusion, we have argued here that, since the very presence of torsion automatically allows parity-violation in the Lagrangian for pure gravity, it should be possible to incorporate the latter in the geometry of space-time itself. We do this by extending the covariant derivative with a set of pseudo-tensorial connections, proportional to the torsion (or more precisely contortion) tensor itself. We have demostrated that the additional terms are uniquely fixed by the condition for preserving metricity. This gives us a consistent prescription to obtain parity-violating effects in the Lagrangians for pure gravity, spin-1 and spin-1/2 fields, as well as new parity-conserving terms in the first two cases. Thus parity-violation– which can be looked upon as the outcome of torsion itself– can be systematically linked to every sector which is conceivably responsible for torsion in space-time.

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References

[1] See, for example, F W Hehl *et al.*, *Rev. Mod Phys.* **48**, 393 (1976)
R M Wald, *General relativity* (University of Chicago Press, Chicago, 1986)
[2] V de Sabbata and C Sivaram, *Spin and torsion in gravitation* (World scientific, Singapore, 1994)
[3] K Horie, hep-th/9409018
[4] K Horie, hep-th/9506049
[5] R Hojman, C Mukku and W Sayed, *Phys. Rev.* **D22**, 1915 (1980)
[6] J Audretsch, *Phys. Rev.* **D24**, 1470 (1991)
[7] R Hammond, *Gen. Relativ. Gravit.* **23**, 1195 (1991)
[8] For early works see, for example, J Leitner and S Okubo, *Phys. Rev.* **136**, B1542 (1964)
see also, for example, L Almeida *et al*, *Phys. Rev.* **D39**, 677 (1989)
J Losseco *et al*, *Phys. Lett.* **A138**, 5 (1989)
G Bisnovatyi–Kogan, *Nuovo. Cimento.* **107**, 357 (2645)
R Aldrovandi *et al*, *Phys. Rev.* **D50**, 2645 (1994)
[9] See, for example, A Lue, L Wang and M Kamionkowski, astro-ph/9812088 and references therein
[10] B Mukhopadhyaya and S Sengupta, hep-th/9811012
[11] B Figuereido, I Soares and J Tiomno, *Class. Quantum Gravit.* **9**, 1593 (1992)