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Collapsing void in a spatially flat Robertson–Walker universe

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Abstract. We consider here a model of the spherical void (or its precursor) containing low density conducting fluid surrounded by a thick spherical shell of radiation embedded in a Robertson–Walker (RW) universe with flat space sections. The underdense region has a metric which is the special case of a solution given by Maiti [1] surrounded by Vaidya metric. We also assume the RW universe to be filled with a perfect fluid with a linear equation of state. The matching conditions indicate that if the time coordinate in each region is future directed then the underdense region appears to go on contracting to a comoving observer in the universe as the latter expands until it disappears. However, if the pressure in the RW universe vanishes, (approximately the present day condition), the underdense region remains static. We have also extended the space-time coordinates of Vaidya metric to the interior of the underdense region as well as the RW universe. It remains to be seen if the region having Vaidya metric disappears earlier than the interior or vice versa.

Keywords. General relativity; cosmology; void.

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1. Introduction

Astronomical observations in the last two decades have indicated the existence of regions of the universe which appear to be empty, called voids (see [2]). Later evidence indicated that the voids are not completely empty but contain gas [3] or dust [4] or dark matter or radiation but are deficient in luminous matter [5, 6]. Many theoretical investigations into the origin and evolution of such voids have been published. (see Bonnor and Chamorro [7] and the references therein). There have been four principal lines of investigations: small perturbations of homogeneous cosmologies, use of the Einstein–Straus vacuole [8], treatment of the boundary of the void by thin wall approximation and matching of the exact solutions of Einstein's equations including Tolman spacetimes with a homogeneous model of the expanding universe on the outside. The present paper is a representative of the last named approach.

It is believed that primeval inhomogeneities (underdense regions) formed in the radiation-filled early universe which later expanded into voids [9]. Even at the very early epochs there was some excess of matter over antimatter. In fact, the ratio of baryon to photon number was 10^{-9}. This is known as baryon asymmetry ([9] p. 159). We consider a period in the early universe after inflation has taken place so that all inhomogeneities have been smoothed out. Still the universe is very hot. This may be ~ 1 second after the big bang ([10] p. 1799). Matter is ionised and is in the plasma state. We know that plasma has a very high electrical as well as thermal conductivity. Since electromagnetic influences are not important on the large scale, we do not consider them here. For simplicity we assume the presence of a single spherical region of matter and radiation whose density is much below the average. For non-GR work on voids, see Sahni and Coles [36] and Padmanabhan [37]. We denote this as region I and call it the core of the void for the purpose of this paper. In our framework in general relativity (which is essentially a macroscopic theory) we cannot consider microscopic phenomena like creation and annihilation of particles. We also do not know the exact nature of dark matter. So we assume that the universe has already become homogeneous and isotropic after the completion of inflation ([9] p. 262, 270). The core is supposed to be embedded in an RW metric with flat space sections (region III). If, however, the rate of expansion of region I is slower than that of region III, a gap may be formed as considered by Harwit [11, 12]. We take the metric (2.1) in region I, which reduces to the RW form when $a \longrightarrow 0$. In this case ξ takes the place of $k/4$ (k being the spatial curvature). When ξ is positive (hence $k > 0$) region I expands at a slower rate than region III. Since region I contains radially flowing radiation, some of it will flow across the boundary to the exterior space. Hence we shall not have vacuum as considered by Harwit ([11] p. 398–399, [12]) but a shell of radiation characterised by Vaidya metric (for a review on the collapse of radiating stars see Bonnor *et al* [13]).

We shall match the first and second fundamental forms in different regions as this is a coordinate independent method and we need not have the same coordinate patch describing all the regions. The matching conditions give the relations between the different coordinate patches on the bounding surfaces. Using this method we may paste together slices of different solutions of Einstein's field equations expressed in different coordinate systems.

In our model the core is a spherical slice cut out of a special case of the space-time solution found by Maiti [1] (region I). The core is surrounded by a thick shell cut out of Vaidya space-time [14] (region II). This combination has been matched with the RW metric with zero spatial curvature. For simplicity we assume the RW universe to be filled with a perfect fluid with a linear relation between pressure p and density ρ so that the scale factor has the form $\sim t^n$.

COBE satellite has shown that the microwave background is highly isotropic (anisotropy is 1 part in 10^5) [15]. This means that the universe was highly isotropic at the time when radiation decoupled from matter. Then how is it that at present we have clusters, superclusters of galaxies and voids in between? So, many cosmologists have the impression that any small anisotropy present at the time of decoupling may have increased subsequently i.e. a small underdense region may have expanded to its present size. In the model of the void, we are going to consider, the reverse happens, i.e. voids initially present tend to disappear with the expansion of the universe. When the universe becomes matter dominated as at present, the void becomes static. But voids which are supposed to originate in the early universe should not be present now. The facts stated above will pose a problem to the theorists unless the result is very much model dependent. However, the possibility of contracting voids in expanding closed dust universes was shown by Chamorro [16, 17].

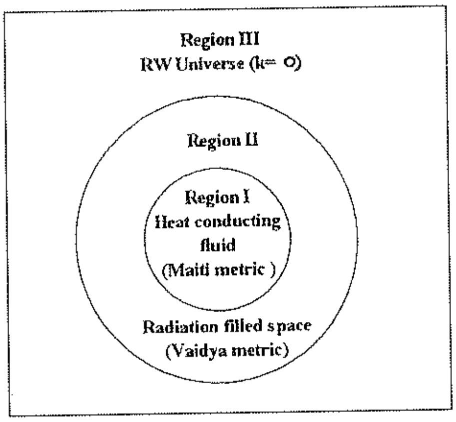

Figure 1. Three regions of spacetime.

In § 2 we present the metrics in the three different regions and the relevant field equations. In § 3 we shall discuss the boundary conditions. In § 4 we discuss the solutions satisfying the above boundary conditions. In § 5 we shall extend the space-time coordinates of Vaidya metric (region II) to regions I and III. In § 6 we shall discuss the main conclusions.

2. Model of the void

The core of the void called region I (figure 1) has a metric of the form given by Maiti [1]:

$$\mathrm{d}s_{\mathrm{I}}^2 = \left[1 + \frac{a}{1+\xi r_{\mathrm{I}}^2}\right]^2 \mathrm{d}t_{\mathrm{I}}^2 - \frac{R^2(t_{\mathrm{I}})}{(1+\xi r_{\mathrm{I}}^2)^2}(\mathrm{d}r_{\mathrm{I}}^2 + r_{\mathrm{I}}^2\mathrm{d}\theta^2 + r_{\mathrm{I}}^2 \sin^2\theta\, \mathrm{d}\phi^2) \tag{2.1}$$

where a and ξ are both constants. The energy momentum tensor is that of a fluid with heat flux expressed in the standard form as

$$T_\mu^\nu = (\rho + p)u_\mu u^\nu - p\delta_\mu^\nu - q_\mu u^\nu - u_\mu q^\nu \tag{2.2}$$

where q^μ represents the heat flux vector which is orthogonal to the velocity vector u^μ. In the present spherically symmetric case the radial component q^1 is nonvanishing. In comoving coordinates Einstein's field equations for the metric (2.1) and energy momentum tensor (2.2) are

$$8\pi p = -\frac{4\xi}{R^2} - \frac{4a\xi}{R^2}\frac{1-\xi r_{\mathrm{I}}^2}{1+\xi r_{\mathrm{I}}^2}\left(1+\frac{a}{1+\xi r_{\mathrm{I}}^2}\right)^{-1} - [2\ddot{R}/R + (\dot{R}/R)^2]\left(1+\frac{a}{1+\xi r_{\mathrm{I}}^2}\right)^{-2} \tag{2.3}$$

$$8\pi\rho = 12\xi/R^2 + 3(\dot{R}/R)^2\left(1 + \frac{a}{1+\xi r_{\mathrm{I}}^2}\right)^{-2} \tag{2.4}$$

$$8\pi q^1 = -\frac{4a\xi r_{\mathrm{I}}\dot{R}}{R^3}\left(1 + \frac{a}{1+\xi r_{\mathrm{I}}^2}\right)^{-2}. \tag{2.5}$$

By taking suitable values of ξ, a and choosing an adequate solution for R we can make p and ρ inside the void small.

In region II the metric is taken in the form of Vaidya [14]:

$$\mathrm{d}s_{\mathrm{II}}^2 = \left(1 - \frac{2m(v)}{r_2}\right)\mathrm{d}v^2 + 2\mathrm{d}v\,\mathrm{d}r_2 - r_2^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{2.6}$$

In Region III we have the RW metric with flat space sections filled with a perfect fluid with $p = \gamma\rho$, $0 \leq \gamma \leq 1/3$. The corresponding scale factor involves a power in t_3:

$$\mathrm{d}s_{\mathrm{III}}^2 = \mathrm{d}t_3^2 - t_3^{2n}(\mathrm{d}r_3^2 + r_3^2\,\mathrm{d}\theta^2 + r_3^2\sin^2\theta\,\mathrm{d}\phi^2) \tag{2.7}$$

where $n = 2/3(\gamma + 1)$.

When $\gamma = 0$, $n = 2/3$ and for $\gamma = 1/3$, $n = 1/2$. (2.7a)

3. Boundary conditions

The matching conditions on the boundary between regions I and II are obtained by equating the first and the second fundamental forms on the two sides. (For a definition, please see Weatherburn [18] pp. 123–128, a detailed discussion of the junction conditions is given in Misner *et al* ([19] pp. 551–554)). The method has been applied to Vaidya metric by Santos [20] and to flat space RW metrics by Dey [21] and Dey and Banerji [22, 23]. We give below the outline of the method to make the paper self-contained.

We consider a 3-space Σ which divides the space-time into two distinct four dimensional manifolds V^+ (interior space time) and V^- (exterior space time). Let g_{ij} be the intrinsic metric to Σ, then

$$\mathrm{d}s_\Sigma^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{3.1}$$

Here latin indices represent the values 1,2,3 and Greek indices the values 1, 2, 3, 4. Equation (3.1) is an invariant known as the first fundamental form. The metrics in $V^\pm$ are given by

$$\mathrm{d}s_\pm^2 = a_{\alpha\beta}^\pm\,\mathrm{d}y_\pm^\alpha\,\mathrm{d}y_\pm^\beta. \tag{3.2}$$

The x^i's are the intrinsic coordinates to the 3-space Σ and $y_\pm^\alpha$'s are the coordinates in $V^\pm$. When approaching Σ in V^+ or V^- we demand that the first fundamental forms of the boundary Σ, $\mathrm{d}s_+^2 = \mathrm{d}s_-^2$. Here

$$g_{ij} = a_{\alpha\beta}y_{,i}^\alpha y_{,j}^\beta \tag{3.3}$$

where $y_{,i}^\alpha \equiv \partial y^\alpha/\partial x^i$.

Let N^α be the unit vector normal to V_3 given by the equations:

$$a_{\alpha\beta}y_{,i}^\alpha N^\beta = 0,\; a_{\alpha\beta}N^\alpha N^\beta = \pm 1. \tag{3.4a, b}$$

The positive sign corresponds to a spacelike hypersurface and the negative sign to a timelike hypersurface. Since the 3-space is necessarily time-like we take the second sign. Equation (3.4a,b) fixes the normal except for the direction. Thus we have two alternative signs for the normal vector at each side of the hypersurface. We must choose these two normal vectors in such a way that if N_α^- points from V^- outwards then N_α^+ points inwards over V^+ and vice versa. This identification determines the arrow of time as we shall see below (see e.g. Fayos *et al* [24], p. 4863).

The extrinsic curvature of Σ is given by

$$\Omega_{ij}^{\pm} = y_{;ij}^{\alpha} a_{\alpha\beta}^{\pm} N_{\pm}^{\beta} \tag{3.5}$$

$$y_{;ij}^{\alpha} = y_{,ij}^{\alpha} + \Gamma_{ij}^{\alpha} y_{,i}^{\beta} y_{,j}^{\gamma} - \Gamma_{ij}^{h} y_{,h}^{\alpha}. \tag{3.6}$$

The Christoffel symbols with Greek indices are formed in 4-space while those with latin indices are formed in 3-space Σ. The invariant

$$\mathrm{d}s_{\pm}^2 = \Omega_{ij}^{\pm} \mathrm{d}x^i \mathrm{d}x^j \tag{3.7}$$

is called the second fundamental form. This should also match on the two sides.

The advantage of the above method is that we may cut out 4-spaces expressed in different coordinate systems and paste them together on a 3-space. The junction conditions then give the relations between the coordinates on the two sides. The papers by Bonnor and Chamorro [7,25,26] take the same coordinate patch to extend over all the regions. It is not necessary for our method.

Let the intrinsic metric on the bounding surface Σ be given by

$$\mathrm{d}s_{\Sigma}^2 = \mathrm{d}\tau^2 - S^2(\tau)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{3.8}$$

Matching the first fundamental forms $\mathrm{d}s_{\mathrm{I}\Sigma}^2 = \mathrm{d}s_{\Sigma}^2 = \mathrm{d}s_{\mathrm{II}\Sigma}^2$ we obtain

$$\left\{1 + \frac{a}{1+\xi r_1^2}\right\}^2 \dot{t}_1^2 - \frac{R^2 \dot{r}_1^2}{(1+\xi r_1^2)^2} = 1 \tag{3.9}$$

$$r_2 = \frac{R(t_1) r_1}{1+\xi r_1^2} \tag{3.10}$$

$$\left\{1 - \frac{2m(v)}{r_2}\right\}\dot{v}^2 + 2\dot{r}_2\dot{v} = 1. \tag{3.11}$$

A dot denotes derivative with respect to τ.

Now we determine the extrinsic curvature tensor of the bounding surface for region I:

$$\Omega_{\theta\theta} = \frac{\Omega_{\phi\phi}}{\sin^2\theta} = \pm\left[\frac{r_1 R(t_1)(1+a+\xi r_1^2)(1+\xi r_1^2)}{(1+\xi r_1^2)^3}\dot{t}_1 + \frac{r_1^2 R^2(t_1) R'(t_1)}{(1+\xi r_1^2)^2(1+a+\xi r_1^2)}\dot{r}_1\right] \tag{3.12}$$

$$\begin{aligned}\Omega_{\tau\tau} = \pm\Bigg[&\frac{(1+a+\xi r_1^2)R(t_1)}{(1+\xi r_1^2)^2)}\ddot{t}_1\dot{r}_1 - \frac{(1+a+\xi r_1^2)R(t_1)}{(1+\xi r_1^2)^2}\ddot{r}_1\dot{t}_1\\ &+\frac{2\xi r_1(1-a+\xi r_1^2)R(t_1)}{(1+\xi r_1^2)^3}\dot{r}_1^2\dot{t}_1 + \frac{2a\xi r_1(1+a+\xi r_1^2)^2}{R(1+\xi r_1^2)^3}\dot{t}_1^3\\ &-\frac{2(1+a+\xi r_1^2)R'}{(1+\xi r_1^2)^2}\dot{r}_1\dot{t}_1^2 + \frac{R^2 R'}{(1+a+\xi r_1^2)(1+\xi r_1^2)^2}\dot{r}_1^3\Bigg]\end{aligned} \tag{3.13}$$

where $R' = \mathrm{d}R/\mathrm{d}t_1$.

The extrinsic curvature tensor of the bounding surface for region II is

$$\Omega_{\theta\theta} = \pm\left[\left(1-\frac{2m(v)}{r_2}\right)\dot{v}r_2 + \dot{r}_2 r_2\right] = \frac{\Omega_{\phi\phi}}{\sin^2\theta} \tag{3.14}$$

$$\Omega_{\tau\tau} = \mp\left[\left(\frac{m\dot{v}}{r_2^2} - \frac{\ddot{v}}{\dot{v}}\right)\right] \tag{3.15}$$

Matching (3.12), (3.13) and (3.14), (3.15) we obtain

$$\frac{r_1 R(t_1)(1+a+\xi r_1^2)(1-\xi r_1^2)}{(1+\xi r_1^2)^3}\dot{t}_1 + \frac{r_1^2 R^2(t_1)R'(t_1)}{(1+\xi r_1^2)^2(1+a+\xi r_1^2)}\dot{r}_1$$
$$= \pm\left[\left(1-\frac{2m(v)}{r_2}\right)\dot{v}r_2 + \dot{r}_2 r_2\right] \tag{3.16}$$

$$\left[\frac{(1+a+\xi r_1^2)R(t_1)}{(1+\xi r_1^2)^2}\ddot{t}_1\dot{r}_1 - \frac{(1+a+\xi r_1^2)R(t_1)}{(1+\xi r_1^2)^2}\ddot{r}_1\dot{t}_1 + \frac{2\xi r_1(1-a+\xi r_1^2)R(t_1)}{(1+\xi r_1^2)^3}\dot{r}_1^2\dot{t}_1\right.$$
$$+\frac{2a\xi r_1(1+\xi r_1^2)^2}{R(1+\xi r)^3}\dot{t}_1^3 - \frac{2(1+a+\xi r_1^2)R'(t_1)}{(1+\xi r_1^2)^2}\dot{r}_1\dot{t}_1^2$$
$$\left.+\frac{R^2(t_1)R'(t_1)}{(1+a+\xi r_1^2)\ (1+\xi r_1^2)}\dot{r}_1^3\right] = \mp\left[\left(\frac{m\dot{v}}{r_2^2} - \frac{\ddot{v}}{\dot{v}}\right)\right]. \tag{3.17}$$

A number of papers have been published on the problem of matching a Vaidya metric with an RW metric, in particular, or with a nonstatic spherically symmetric metric, in general [27, 28, 24]. Fayos *et al* ([28] p. 2735) write: "There are metrics which are usually thought of as nonradiating, and therefore one might think that they should not be matchable to Vaidya exterior. The fact that they are matchable indicates that there are alternative interpretations for the interior metrics in which they do radiate." These remarks also apply to our case of the RW metric. We shall discuss this more fully at the end of § 4.

Similarly matching the first fundamental forms on the bounding surface of regions II and III we obtain two equations

$$\dot{t}_3^2 - t_3^{2n}\dot{r}_3^2 = 1 \tag{3.18}$$

$$r_2 = r_3 t_3^n \tag{3.19}$$

and again eq. (3.10) on the other surface.

Matching the second fundamental forms we obtain

$$r_3 t_3^n \dot{t}_3 + n r_3^2 \dot{r}_3 t_3^{3n-1} = \pm\left[\left(1-\frac{2m(v)}{r_2}\right)\dot{v}r_2 + \dot{r}_2 r_2\right] \tag{3.20}$$

$$\ddot{r}_3\dot{t}_3 t_3^n + 2n\dot{r}_3\dot{t}_2^2 t_3^{n-1} - n\dot{r}_3^3 t_3^{3n-1} - \dot{r}_3 t_3^n \ddot{t}_3 = \pm\left[\left(\frac{m\dot{v}}{r_2^2} - \frac{\ddot{v}}{\dot{v}}\right)\right]. \tag{3.21}$$

4. Solutions of the matching conditions

4.1 *Boundary between regions I and II*

We have now to find the solutions of the matching conditions (3.9)–(3.11),(3.16),(3.17). Let us assume the boundary of regions I and II to be stationary in the comoving coordinates of region I, i.e.

$$(r_1)_\Sigma = r_0. \tag{4.1}$$

From eq. (3.8) we obtain

$$\dot{t}_1 = \frac{1+\xi r_0^2}{1+a+\xi r_0^2} \tag{4.2}$$

taking only the positive sign. We assume that t_1 increases when τ does so. Equations (3.16) and (3.17) become on using (4.2):

$$\frac{r_0(1-\xi r_0^2)R(t_1)}{(1+\xi r_0^2)^2} = \pm\left[\left(1-\frac{2m(v)}{r_2}\right)\dot{v}r_2 + \dot{r}_2 r_2\right] \tag{4.3}$$

$$\frac{2a\xi r_0}{(1+a+\xi r_0^2)R(t_1)} = \mp\left[\left(\frac{m\dot{v}}{r_2^2} - \frac{\ddot{v}}{v}\right)\right]. \tag{4.4}$$

Substituting the value of r_2 from eq. (3.10) in eq. (3.11) and solving the quadratic equation we obtain

$$\dot{v} = \frac{-\frac{R' r_0}{1+a+\xi r_0^2} \pm \sqrt{\frac{R'^2 r_0^2}{(1+a+\xi r_0^2)^2} + \left\{1 - \frac{2m(1+\xi r_0^2)}{R r_0}\right\}}}{\left\{1-\frac{2m(1+\xi r_0^2)}{R r_0}\right\}}. \tag{4.5}$$

Solving for $\dot{v}$ in (4.3) we obtain

$$\dot{v} = \frac{-\frac{R' r_0}{1+a+\xi r_0^2} \pm \frac{1-\xi r_0^2}{1+\xi r_0^2}}{\left\{1-\frac{2m(1+\xi r_0^2)}{R r_0}\right\}}. \tag{4.6}$$

Comparing (4.5) and (4.6) and squaring we obtain

$$\frac{4\xi r_0^2}{(1+\xi r_0^2)^2} = \frac{2m(1+\xi r_0^2)}{R r_0} - \frac{R'^2 r_0^2}{(1+a+\xi r_0^2)^2}. \tag{4.7}$$

Differentiating both sides with respect to t_1 we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t_1}(m/R) = \frac{r_0^3 R'' R'}{(1+\xi r_0^2)(1+a+\xi r_0^2)^2}. \tag{4.8}$$

A simple solution of this was found by Banerjee *et al* [29]. When $R''=0$, i.e.

$$R = bt_1,\ b \text{ is a constant.} \tag{4.9}$$

Here we have chosen the origin of time t_1 in such a way that the additive constant vanishes, such that $R \to 0$ when $t_1 \to 0$.

In this case

$$m/R = c(\text{constant}).$$

Using (4.9) we obtain

$$m = bct_1 \tag{4.10}$$

Here m must be positive. If region I is collapsing then t_1 increases from negative values towards zero, then b and c should be of opposite signs and m decreases with time. Using (4.6) and (4.2) we obtain

$$\mathrm{d}v/\mathrm{d}t_1 = \dot{v}/\dot{t}_1 = \frac{(1+a+\xi r_0^2)(1-\xi r_0^2) - br_0(1+\xi r_0^2)}{(1+\xi r_0^2)^2\left\{1 - \frac{2c(1+\xi r_0^2)}{r_0}\right\}}. \tag{4.11}$$

Equations (4.11) and (4.2) show that on Σ, v is a linear function of t_1 and τ. The denominator is positive but the numerator may be positive or negative.

Substituting from (4.9) and (4.10) in (4.7) we obtain

$$c = \frac{2\xi r_0^3}{(1+\xi r_0^2)^3} + \frac{b^2 r_0^3}{2(1+a+\xi r_0^2)^2(1+\xi r_0^2)}. \tag{4.12}$$

If a, ξ are known then the above equation gives c in terms of the radius r_0 of region I, and b.

We have seen above that $\dot{v}$ is a constant. Hence $\ddot{v}$ vanishes. So eq. (4.4) gives

$$\frac{m\dot{v}}{r_2^2} = \mp \frac{2a\xi r_0}{R(t_1)(1+a+\xi r_0^2)}. \tag{4.13}$$

Substituting for $\dot{v}$, m and r_2 from (4.6), (4.10), and (3.10–3.11) we obtain another equation connecting c, b and r_0. Thus we obtain both b and c in terms of r_0.

Let us now introduce t_2 on Σ by the equation

$$v = \pm t_2 - \int \frac{\mathrm{d}r_2}{1 - \frac{2m}{r_2}}. \tag{4.14}$$

The second sign corresponds to advanced time going towards the past. Taking the first sign and substituting the values of r_2 and m we obtain using (4.11):

$$\begin{aligned} t_2 &= v + t_1 \frac{\frac{br_0}{1+\xi r_0^2}}{\left\{1 - \frac{2c(1+\xi r_0^2)}{r_0}\right\}} \\ &= t_1 \frac{(1+a+\xi r_0^2)(1-\xi r_0^2)}{(1+\xi r_0^2)^2\left\{1 - \frac{2c(1+\xi r_0^2)}{r_0}\right\}}. \end{aligned} \tag{4.14a}$$

The sign should be chosen in such a way that when t_2 increases, t_1 also increases.

4.2 *Boundary between regions II and III*

If the boundary between regions II and III appears static to an observer in comoving coordinates in region III, $r_3 = \alpha_0$ (constant), $\dot{r}_3 = 0$, then the boundary conditions (3.11), (3.18)–(3.21) give after some manipulations:

$$\dot{t}_3 = 1 \text{ (taking only the positive sign)} \tag{4.15}$$

$$r_2 = \alpha_0 t_3^n \tag{4.16}$$

$$\mathrm{d}v/\mathrm{d}t_3 = \frac{-n\alpha_0 t_3^{n-1} \pm 1}{1 - \frac{2m}{\alpha_0} t_3^{-n}} \tag{4.17}$$

$$2m = n^2 \alpha_0^3 t_3^{3n-2} \tag{4.18}$$

$$\frac{m\dot{v}}{r_2^2} - \frac{\ddot{v}}{\dot{v}} = \frac{\frac{n\alpha_0}{2}(3n-2)t_3^{n-2}}{1 + n\alpha_0 t_3^{n-1}} = 0. \tag{4.19}$$

Equation (4.19) can be satisfied only when $n = 2/3$ which corresponds to $\gamma = 0$ (eq. 2.7a) i.e. a matter dominated universe where the pressure can be neglected. This shows that the boundary between regions II and III can remain static only when our universe is matter dominated. In all other cases the bounding surface moves. Further, when $n = 2/3$ eq. (4.18) shows that m is constant and Vaidya metric reduces to that of Schwarzschild:

$$\mathrm{d}s^2 = (1 - 2m/r_2)\mathrm{d}t_2^2 - (1 - 2m/r_2)^{-1}\mathrm{d}r_2^2 - r_2^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{4.20}$$

under the transformation

$$v = t_2 - \int \frac{\mathrm{d}r_2}{1 - 2m/r_2}.$$

Substituting from (4.17), (4.18) and (4.16) we obtain

$$t_2 = \pm \int \frac{\mathrm{d}t_3}{1 - 4/9\alpha_0^2 t_3^{-2/3}}$$

$$= \left[t_3 + 4/3\alpha_0^2 t_3^{1/3} + 4/9\alpha_0^3 \ln\left| \frac{3t_3^{1/3} - 2\alpha_0}{3t_3^{1/3} + 2\alpha_0} \right| \right]. \tag{4.21}$$

This agrees with the expression found by Dey and Banerji [22]. If we want both t_2 and t_3 to be future directed, we must take the first sign and reject the second.

Let us assume

$$r_3 = u_3 = \alpha t_3^a + \alpha_0 \tag{4.22}$$

where α, a and α_0 are constants.

Taking only the positive sign of the square root, we obtain from (3.18)

$$\dot{t}_3 = [1 - \alpha^2 a^2 t_3^{2(n+a-1)}]^{-1/2}. \tag{4.23}$$

From (3.19) we obtain

$$r_2 = u_3 t_3^n = \alpha t_3^{n+a} + \alpha_0 t_3^n. \tag{4.24}$$

From (3.11) we obtain

$$\dot{v} = \frac{-\dot{r}_2 \pm \sqrt{\dot{r}_2^2 + (1 - 2m/r_2)}}{(1 - 2m/r_2)}. \tag{4.25}$$

From (3.20) and (4.25) we obtain

$$2m = n^2 u_3^3 t_3^{3n-2}. \tag{4.26}$$

We substitute all the quantities in eq. (3.21) and find under what conditions it is satisfied. The condition is

$$a = 1 - n \tag{4.27}$$

and

$$\pm 3n^2(1-n)\alpha^2 - n\alpha \mp (2-3n) = 0. \tag{4.28}$$

The first sign gives

$$\alpha = 1/n \text{ or } = \frac{3n-2}{3n(1-n)}. \tag{4.29}$$

Case I

$$u_3 = 1/nt_3^{1-n} + \alpha_0. \tag{4.30}$$

From (4.23) we obtain

$$\dot{t}_3 = [1 - (1-n)^2/n^2]^{-1/2} \tag{4.31}$$

when $n \to 1/2$, $\dot{t}_3 \to \infty$. Hence, in the early universe we have always $\mathrm{d}\tau \to 0$ for $\mathrm{d}t_3 \neq 0$. So we reject this solution.

Case II

$$u_3 = \alpha_0 - \frac{2-3n}{3n(1-n)} t_3^{1-n} \qquad (\alpha_0 > 0). \tag{4.32}$$

There is no restriction on the value of α_0 which depends naturally on initial conditions. For $1/2 \leq n \leq 2/3$ the value of u_3 gradually decreases when t_3 increases. This shows that the void contracts when the universe expands. The result was found by other methods by Fayos *et al* [26] [eqs. after their eq. (51)].

The second sign gives

$$\alpha = -1/n \text{ or } \frac{2-3n}{3n(1-n)}. \tag{4.33}$$

The first value leads to inconsistency as before. The second value gives an expanding void but here t_2 is past directed when t_3 is future directed as shown above in eq. (4.21). Therefore, this value should be rejected.

Using (4.32) we have from (3.19)

$$r_2 = u_3 t_3^n = \alpha_0 t_3^n - \frac{2-3n}{3n(1-n)} t_3. \tag{4.34}$$

From (4.23) and (4.27) we obtain

$$\dot{t}_3 = \frac{3n}{2(3n-1)^{1/2}}. \tag{4.35}$$

From (4.25), (4.26), (4.32), (4.34) and (4.35) we obtain

$$v = \{2(1-n)/n\} \int \frac{\mathrm{d}t_3}{3n(1-n)\alpha_0 t_3^{n-1} + 1}. \tag{4.36}$$

Using (4.14) we obtain

$$t_2 = \{3\alpha_0(1-n)\}^{1/1-n}(n)^{n/1-n}\left[(3n-1)\int \frac{x^{(2-n)/(n-1)}\mathrm{d}x}{x-(5-6n)} - \int \frac{x^{(2-n)/(n-1)}\mathrm{d}x}{1+x}\right] \tag{4.37}$$

where $x = 3n(1-n)\alpha_0 t_3^{n-1}$.

From (4.26) and (4.32) we obtain

$$\frac{\mathrm{d}m}{\mathrm{d}t_3} = -\frac{n(2-3n)^{u_3^2}}{2} t_3^{2(n-1)}(1 + nu_3 t_3^{n-1}). \tag{4.38}$$

This is negative for $1/2 \leq n < 2/3$ and zero for $n = 2/3$.

We have seen above that in our model the boundary between regions II and III is not comoving with the matter of the RW universe in the normal perfect fluid interpretation where the energy momentum tensor is

$$T_{\mu\nu} = (\bar{\rho} + \bar{p})u_\mu u_\nu - \bar{p}g_{\mu\nu}. \tag{4.39}$$

Thus an accretion process occurs on to the region III [27].

Tupper [30] remarked as follows: "Given a metric tensor that does, in fact, lead to a $T_{\mu\nu}$ satisfying the energy conditions, how do we know what type of field this energy tensor represents? Put another way, is it possible that $T_{\mu\nu}$ is nonuniquely a viable energy tensor, i.e. can the energy tensors of two apparently different fields be identical in that they have precisely the same components? The answer to this is in the affirmative". Tupper and his coworkers gave a number of examples (e.g. [31, 32, 33]).

Fayos *et al* [27] showed how to construct a $T_{\mu\nu}$ in which the cosmological fluid is comoving with Σ having the same velocity v_μ and additional terms as follows:

$$T_{\mu\nu} = (\rho + p)v_\mu v_\nu - pg_{\mu\nu} - \Omega^2 l_\mu l_\nu + \Pi_{\mu\nu}. \tag{4.40}$$

Here $v^0 = v_0 = 3n/2(3n-1)^{1/2}$, $v^1 = -\{(2-3n)/2(3n-1)^{1/2}\}t_3^{-n}$,

$$v_1 = \{(2-3n)/2(3n-1)^{1/2}\}t_3^n \tag{4.41}$$

l^μ is an outgoing radial null vector field (the minus sign indicates that the radiation is incoming in region III), and $\Pi_{\mu\nu}$ is a tensor of anisotropic pressures, which is tracefree and orthogonal to v^μ. The values of $T_{\mu\nu}$ given by (4.39) and (4.40) are identical. We take on the boundary Σ:

$$\begin{aligned} p &= \bar{p} + \{(2-3n)/6(3n-1)\}(\bar{\rho}+\bar{p}) = \{(2-3n)/6(3n-1)\}\bar{\rho} \\ &\quad + \{(15n-4)/6(3n-1)\}\bar{p}, \\ \rho &= \bar{\rho} + \{(2-3n)/2(3n-1)\}(\bar{\rho}+\bar{p}) = \{3n/2(3n-1)\}\bar{\rho} \\ &\quad + \{(2-3n)/2(3n-1)\}\bar{p}, \quad l_0 = 1, l_1 = t_3^n, \end{aligned}$$

$$\Omega^2=\{3n(2-3n)/4(3n-1)^2\}(\bar{\rho}+\bar{p}),\ \ \Pi_{00}=\{(2-3n)^3/12(3n-1)^2\}(\bar{\rho}+\bar{p}),$$
$$\Pi_{01}=\Pi_{00}\cdot 3nt_3^n/(2-3n),\quad \Pi_{11}=\Pi_{00}.9n^2t_3^{2n}/(2-3n)^2,$$
$$\Pi_{22}=-\Pi_{00}\cdot 2(3n-1)u_3^2t_3^{2n}/(2-3n)^2,\quad \Pi_{33}=\Pi_{22}\sin^2\theta. \tag{4.42}^*$$

Obviously for $n=2/3(\gamma=0)$ (4.39) and (4.40) are identical.

$$\text{At a distance from } \Sigma, v^1=f/(1-f^2t_3^{2n})^{1/2} \tag{4.43}$$

where $f(t_3, r_3)$ is an arbitrary function subject to the restriction

$$f\{t_3, u_3(t_3)\}=\mathrm{d}u_3/\mathrm{d}t_3=-\{(2-3n)/3n\}t_3^{-n}. \tag{4.44}$$

In normal situations the velocity of Σ, $|\mathrm{d}u_3/\mathrm{d}t_3|$ is very small and we can always choose f in such a way that it decreases monotonically away from Σ. So, at some distance from the boundary of the void there is very little departure of the RW universe from the usual perfect fluid interpretation. A comoving observer there will find the void to be collapsing. As the radiation enters region III, it can be scattered or absorbed by the matter existing there. The different possibilities of how this radiation gradually fades away can be described by the different choices of the arbitrary function f.

From (4.42) we find that

$$\rho+p=\{(3n+1)/3(3n-1)\}(\bar{\rho}+\bar{p}). \tag{4.45}$$

Here $1/2\le n\le 2/3$, so $\rho+p>0$ for $(\bar{\rho}+\bar{p})>0$. For the energy momentum tensor given by (4.39) and metric (2.7) Einstein's field equation gives

$$\bar{\rho}=n^2/t^2>0.$$

Further $\bar{p}=\gamma\bar{\rho}\ge 0$ because $0\le\gamma\le 1/3$. This means that the strong energy condition is obeyed. So, we find that the energy momentum tensor of (4.40) corresponds to that of reasonable matter of region III.

We have seen above that a comoving observer in region I finds the boundary between regions I and II to be static while a comoving observer in region III, a little away from the boundary Σ of the void finds Σ to be collapsing. We may define a comoving observer in the present day universe to be one to whom the microwave background radiation has no dipole anisotropy ([9] p. 380, [34]). According to such an observer the boundary between regions II and III contracts with time. Now the question arises as to whether the region II will ever disappear or not. This can be determined by expressing the two boundaries in terms of r_2, t_2 coordinates. We may also like to know the behaviour of bounding surfaces as seen by an observer somewhere within region III or region I. Such questions can be answered if we extend the same coordinates to both the regions. With this end in view we have extended the coordinate patch in region II to the interior of regions I and III in the following section.

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* There is a misprint in the expression for Π_{22} in ref. [27]. The correct expression is obtained by replacing x by x^2 in both the numerator and denominator.

5. Extension of coordinates r_2 and t_2 to regions I and III

5.1 *Region I*

We may extend the radial coordinates r_2 to the interior of region I by using eq. (3.9) and (4.9).

$$r_2 = \frac{br_1 t_1}{1+\xi r_1^2}. \tag{5.1}$$

We extend the coordinate t_2 by requiring it to be orthogonal to r_2. Hence (see de la Cruz [35] for the method)

$$\frac{\partial t_2/\partial t_1}{\partial t_2/\partial r_1} = \frac{(RR')^{-1}}{\frac{r_1(1+\xi r_1^2)}{(1-\xi r_1^2)(1+a+\xi r_1^2)^2}} = \frac{(b^2 t_1)^{-1}}{r_1(1+\xi r_1^2)(1-\xi r_1^2)^{-1}(1+a+\xi r_1^2)^{-2}}. \tag{5.2}$$

Using (4.14a) we obtain

$$t_2 = \pm e^{b^2 z_1} \frac{(1+a+\xi r_0^2)(1-\xi r_0^2)}{(1+\xi r_0^2)^2\left\{1-\frac{2c(1+\xi r_0^2)}{r_0}\right\}} \tag{5.3}$$

where

$$z_1 = b^{-2}\ln t_1 + 1/[(a+2)^2\xi]\ln\frac{(1+a+\xi r_1^2)(1-\xi r_0^2)}{(1-\xi r_1^2)(1+a+\xi r_0^2)} + \frac{a(r_0^2-r_1^2)}{2(a+2)(1+a+\xi r_1^2)(1+a+\xi r_0^2)}. \tag{5.3a}$$

5.2 *Region III*

In the same way we may extend the radial coordinate r_2 to the interior of region III by using (3.19)

$$r_2 = r_3 t_3^n. \tag{5.4}$$

We take the coordinate t_2 to be orthogonal to r_2 so that

$$\frac{\partial t_2/\partial t_3}{\partial t_2/\partial r_3} = \frac{(1/n)t_3^{1-2n}}{r_3}. \tag{5.5}$$

We may obtain t_2 within region III by replacing x in (4.37) by

$$x = 3n(1-n)\alpha_0\left[Z_2 + \frac{3n(1-n)(2-3n)}{4-3n}\alpha_0\right]^{-1} \tag{5.6}$$

where

$$Z_2 = \left[\left\{t_3^{1-n} - \frac{3n(1-n)(2-3n)}{4-3n}\alpha_0\right\}^2 + \frac{9n^2(1-n)^2}{4-3n}(r_3^2-u_3^2)\right]^{1/2}. \tag{5.6a}$$

6. Conclusion

In this model of the spherical void or its precursor the radius of the void (a combination of regions I and II) formed in the early universe goes on contracting as the universe expands. This is contrary to the belief among some cosmologists that a small anisotropy present at the time of decoupling of matter from radiation increased subsequently. As the universe evolved, various particle species became dominant. Then at temperatures 10^{10}–$10^9 e^{\pm}$ pairs annihilated and raised photon gas temperature above that of neutrinos. Finally at $T \sim 4000\,\mathrm{K}$ recombination took place and matter decoupled from radiation. Now it has become matter-dominated and if a void is formed now or if any such relic is left over from the past, it will remain static. However, in the present framework we cannot find a mathematical solution representing the complete evolution of the universe from the early epoch as particle creation processes cannot be incorporated in this formalism. If we start with a particular value of γ in the equation of state $p = \gamma\rho$ in our formalism, we cannot change its value in the course of evolution. But we know that in the real universe γ has changed from $1/3$ to 0.

The coordinates in region II have been extended to the interiors of regions I and III. It will be interesting to see if the outer shell of the void (region II) vanishes before its core (region I) or vice versa and what the matching conditions would indicate after that. Work is proceeding along these lines. It is interesting also to find out how the bounding surfaces will behave as seen by an observer at some point within region I or III.

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References

[1] S R Maiti, *Phys. Rev.* **D25**, 2518 (1982)
[2] R P Kirshner, A Oemler Jr, P L Schechter and S A Shectman, *Astrophys. J.* **248**, L57 (1981)
[3] N Brosch and P M Gondhalekar, *Astron. Astrophys.* **140**, L43 (1984)
[4] D Lindley, *Nature* (London) **337**, 505 (1989)
[5] V A Balzano and D W Weedman *Astrophys. J.* **255**, L1 (1982)
[6] N Sanduleak and P Pesch, *Astrophys. J.* **258**, L11 (1982)
[7] W B Bonnor and A Chamorro, *Astrophys. J.* **361**, 21 (1990)
[8] A Einstein and E G Straus, *Rev. Mod. Phys.* **17**, 120 (1945)
[9] E W Kolb and M S Turner, *The early universe* (Addison Wesley, 1990)
[10] J M M Senovilla, *Phys. Rev.* **D53**, 1799 (1996)
[11] M Harwit, *Astrophys. J.* **392**, 394 (1992)
[12] M Harwit, *Astrophys. J.* **447**, 482 (1995)
[13] W B Bonnor, A K G de Oliveira and N O Santos, *Phys. Rept.* **181**, 269 (1989)
[14] P C Vaidya, *Nature (London)* **171**, 260 (1953)
[15] G F Smoot, C L Bennett, A Kogut, E L Wright, J Aymon, N W Boggess, E S Cheng, G de Amici, S Gulkio, M G Hauser, G Hinshaw, P D Jackson, M Janssen, E Kaita, T Kelsall, P Keegstra, C Lineweaver, K Loewenstein, P Lubin, J Mather, S S Mayer, S H Moseley,

T Murdoch, L Rokke, R F Silverberg, L Tenorio, R Weiss and D T Wilkinson, *Astrophys. J.* **396**, L1 (1992)
[16] A Chamorro, Proc. 13th Int. Confce. on GRG, Cordoba, Argentina, 28 June – 4 July 1992 (ref. supplied by referee).
[17] A Chamorro, Proc. Second William Fairbank Confce. Hongkong, 13–16 Dec. 1993 (ref. supplied by referee).
[18] C E Weatherburn, An Introduction to Riemannian Geometry and the Tensor Calculus (Cambridge University Press, 1957)
[19] C W Misner, K S Thorne and J A Wheeler, *Gravitation* (Freeman, 1973)
[20] N O Santos, *Mon. Not. R. Astron Soc.* **216**, 403 (1985)
[21] T K Dey, Ph.D. Thesis (unpublished), Burdwan University (1994)
[22] T K Dey and S Banerji, *Phys. Rev.* **D44**, 325 (1991)
[23] T K Dey and S Banerji, *Phys. Rev.* **D48**, 3478 (1993)
[24] F Fayos, J M M Senovilla and R Torres, *Phys. Rev.* **D54**, 4862 (1996)
[25] W B Bonnor and A Chamorro, *Astrophys. J.* **378**, 461 (1991)
[26] A Chamorro, *Astrophys. J.* **383**, 51 (1991)
[27] F Fayos, X Jaen, E Llanta and J M M Senovilla, *Class. Q. Gravit.* **8**, 2057 (1991)
[28] F Fayos, X Jaen, E Llanta and J M M Senovilla, *Phys. Rev.* **D45**, 2732 (1992)
[29] A Banerjee, S B Dutta Choudhury and B K Bhui, *Phys. Rev.* **D40**, 670 (1989)
[30] B O J Tupper, *J. Math. Phys.* **22**, 2666 (1981)
[31] A A Coley and B O J Tupper, *Astrophys. J.* **271**, 1 (1983)
[32] A A Coley and B O J Tupper, *Phys. Lett.* **A95**, 357 (1983)
[33] A A Coley and B O J Tupper, *Astrophys. J.* **280**, 26 (1984)
[34] M Bensadoun, M Bersanelli, G de Amici, A Kogut, S M Levin, M Limon, G F Smoot and C Witebsky, *Astrophys. J.* **419**, 1 (1993).
[35] V de la Cruz, *Phys. Lett.* **22**, 152 (1966)
[36] V Sahni and Coles, *Phys. Rep.* **262**, 1 (1995)
[37] T Padmanabhan, *Structure formation in the universe* (Cambridge University Press, 1993)