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Einstein equations and the joining of discontinuous metrics

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Abstract. We consider the problem of joining of metrics when these are not continuous across the joining (hyper-) surface. We confine ourselves to static, spherically symmetric metrics which join without requiring gradients of a δ-function in the energy-momentum tensor. It is found that a surface tension is always associated in cases where the metrics are discontinuous. In some cases, the joined metrics satisfy Einstein equations (in the sense of distributions) while, in others, the surface tension associated with the limiting discontinuous metric depends on the interpolating functions used to produce it suggesting sensitivity to short distance effects beyond general relativity.

Keywords. Discontinuous metrics; surface tension; Einstein equations.

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1. Introduction

In this note, we shall explore some aspects of the problem of joining metrics satisfying Einstein equations in the interior of two open four dimensional regions V^+ and V^- whose closure satisfies $\bar{V}^+ \cup \bar{V}^- = R^4$ and $\bar{V}^+ \cap \bar{V}^-$ = a three dimensional hypersurface Σ. If the corresponding metrics are called g^+ and g^-, then the joined metric, defined symbolically by $g^+ \cup g^-$, may not satisfy Einstein equations on Σ unless special conditions, called junction conditions, are satisfied thereabout. The problem is trivially solved if the second derivatives of the metric remain continuous across Σ, for, then, Einstein equations hold in the ordinary sense throughout R^4 with a continuous energy-momentum tensor. If only the first derivatives of the metric tensor are continuous, the metrics still join without producing surface singularities (of the δ-function type) in the energy-momentum tensor on Σ but the energy-momentum tensor may be discontinuous on Σ. If, however, the latter has surface singularities, the situation is more complicated. Generally speaking, if the energy-momentum tensor does not possess surface singularities on Σ corresponding to the derivatives of a δ-function, the metric tensor must be continuous; in other words, a discontinuous metric normally implies surface singularities of the gradient of δ-function type. However, there are exceptions and it is these that we want to discuss here.

We shall, for simplicity, confine our attention to static spherically symmetric metrics of the form

$$ds^2 = A\,dt^2 - B\,dR^2 - R^2(\sin^2\theta d\Phi^2 + d\theta^2). \qquad (1)$$

We shall examine situations in which the coefficients $A = g_{00}$ and $B = -g_{11}$ may be discontinuous. (We should mention that the problem of joining metrics is an old one and has been looked at recently by the "early universe" workers [1, 2]. However, our approach and examples are complementary to those of these workers.) An example in which B is discontinuous is easy to construct and simple to interpret. It is the time-honoured model of an infinitely thin energy-shell familiar from nonrelativistic potential theory. (A reference is made to this example in Landau and Lifshitz [3] but our solution is clearer physically). This solution is not very interesting physically; it is being included here to make plausible the appearance of discontinuities in metrics. This solution will be constructed in a mathematically precise way as the limit of a sequence of smooth (in fact C^{∞}) solutions and it will be shown that the limiting solution satisfies Einstein equation with an energy-momentum tensor which has δ-function singularities on a certain surface. The corresponding energy-momentum tensor will be exhibited as a δ-function which is a legitimate distribution (i.e. it does not depend on the form of the precise functions used in constructing the intermediate sequences.) We shall see that this is a two parameter solution depending on the total gravitational mass M and the "radius" R_0 of the shell with $R_0 > 2GM/c^2$. (Henceforth, we shall put $c = 1$ for the velocity of light.)

The other model that we shall consider is one in which A may be discontinuous. This has a greater significance in general for while the quantity B does not have a direct significance in the nonrelativistic (Newtonian) limit, the quantity A is, essentially, the gravitational potential in that limit and obeys Poisson's equation. A discontinuity in A can, in that limit, only occur at mass dipole layers which do not exist. We shall interpret the absence of mass dipole layers to mean that the energy-momentum tensor does not possess singularities involving gradients of a δ-function (though it may possess δ-function singularities not involving gradients or higher derivatives). This is a simplifying assumption and is physically reasonable since such (gradient of δ-function) singularities do not occur elsewhere in this context while δ-function singularities (without gradients) occur, for example, in surface tension. Then we can show, directly and simply from Einstein equations, that a discontinuity in A can occur only for those values of the radial co-ordinate R for which $1/B = 0$ i.e. on a null surface. The simplest example where such null surfaces occur is in (Schwarzschild) black holes. (To avoid confusion we shall state that we call black hole any solution of Einstein equation, *or generalizations thereof*, in which matter is confined within its event horizon at $R = 2\,GM$ where M is the mass of the black hole). Thus we shall try to construct a discontinuous solution (in A) which may be relevant to black holes. Such a solution will be obtained by joining an Einstein model universe solution to the external Schwarzschild solution. This problem was considered elsewhere [4]. We shall add that the problem has some curiosity value even if it is not physically relevant. The solution will be constructed, as above, by taking the limit of a sequence of continuous metrics and finding the energy-momentum tensor in the limit. This solution is somewhat unusual in that it corresponds to a potentially infinite surface tension whose pre-limiting value depends essentially on the nature of functions used to construct the approximating sequence; it does not, therefore, satisfy Einstein equation in the sense of distributions.

The plan of the paper is as follows: In § 2, we consider our first model in which B is discontinuous. In § 3, we consider our main model in which A is discontinuous. In § 4 we state our conclusions and end with some comments.

2. Thin spherical shell

The gravitational potential of a thin shell being well known, it is convenient to choose to describe the quantity A of a spherically symmetric distribution of energy corresponding to a given total mass M by

$$A = 1 - \frac{R_g}{R}\int_0^R x^2 f(x)\,\mathrm{d}x - R_g\int_R^\infty x f(x)\,\mathrm{d}x. \tag{2}$$

We shall normalize the class of functions $f(x)$ by

$$\int_0^\infty x^2 f(x)\,\mathrm{d}x = 1, \tag{3}$$

then $R_g = 2\,GM$ is the Schwarzschild radius of the object. We shall assume that $f(x)$ is sharply peaked at $R = R_0$ (which will be kept fixed). (It turns out that our equations make sense only for $R_0 > R_g$). We shall for definiteness also assume that

$$\int_0^\infty x f(x)\,\mathrm{d}x = \frac{1}{R_0}. \tag{4}$$

With this choice, in the limit of a thin shell, our A becomes

$$A = \left(1 - \frac{R_g}{R}\right)\Theta(R - R_0) + \left(1 - \frac{R_g}{R_0}\right)\Theta(R_0 - R) \tag{5}$$

and is continuous. Einstein equations for this textbook case are (with $G^v_\mu = R^v_\mu - \frac{1}{2}R\delta^v_\mu$)

$$G^0_0 = \frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left[R\left(\frac{1}{B} - 1\right)\right] = -8\pi G T^0_0, \tag{6}$$

$$G^1_1 = \frac{A'}{RAB} + \frac{1}{R^2}\left(\frac{1}{B} - 1\right) = -8\pi G T^1_1, \tag{7}$$

$$G^2_2 = G^3_3 = \frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{A'}{2AB}\right] + \frac{A'^2}{4A^2B} - \frac{A'}{4A}\left(\frac{1}{B}\right)' + \frac{A'}{2RAB} + \frac{1}{2R}\left(\frac{1}{B}\right)' = -8\pi G T^2_2 = -8\pi G T^3_3. \tag{8}$$

We anticipate δ-function type contributions to T^0_0, T^2_2 and T^3_3. However, T^1_1 cannot have any δ-function type contribution (in the limit of a thin shell). This follows because $1/B$ is not differentiated in (7). Then, in the limit of a thin shell, $T^1_1 = 0$ and so, for convenience, we let $T^1_1 \equiv 0$. This determines B and we get

$$\frac{1}{B} = 1 - \left[\frac{R_g}{R}\int_0^R x^2 f(x)\,\mathrm{d}x \Big/ \left\{1 - R_g\int_R^\infty x f(x)\,\mathrm{d}x\right\}\right]. \tag{9}$$

Note that

$$\frac{1}{B} = \left(1 - \frac{R_g}{R}\right)\Theta(R - R_0) + \Theta(R_0 - R), \tag{10}$$

in the limit of a thin shell and is discontinuous. With A and B known, one can calculate T_μ^v. It is easily shown that

$$T_0^0 = \frac{M}{4\pi R_0^2}\delta(R - R_0), \tag{11}$$

in the limit of a thin shell. This calculation is typical of such calculations later so we exhibit it here. Let $\Psi \in C_0^\infty(0,\infty)$ (any infinitely differentiable fixed function of compact support over $(0,\infty)$). Then

$$\begin{aligned}\int_0^\infty \Psi(R) T_0^0\,\mathrm{d}R &= -\frac{1}{8\pi G}\int_0^\infty \mathrm{d}R\frac{\Psi(R)}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\left(\frac{1}{B}-1\right)\right)\\ &= \frac{1}{8\pi G}\int_0^\infty \mathrm{d}R\left(\frac{1}{B}-1\right)\left[\frac{\Psi'(R)}{R}-\frac{2\Psi(R)}{R^2}\right]\\ &\to \frac{-R_g}{8\pi G}\int_{R_0}^\infty \left[\frac{\Psi'(R)}{R^2}-\frac{2\Psi(R)}{R^3}\right]\mathrm{d}R\\ &= -\frac{M}{4\pi}\int_{R_0}^\infty \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{\Psi(R)}{R^2}\right)\mathrm{d}R = \frac{M}{4\pi R_0^2}\Psi(R_0),\end{aligned} \tag{12}$$

in the limit of a thin shell. Similar calculation gives

$$T_2^2 = T_3^3 = \frac{\delta(R-R_0)}{32\pi G R_0}\left(\frac{R_g}{R_0} + \ln\left(1-\frac{R_g}{R_0}\right)\right). \tag{13}$$

It is instructive to check this result directly by using the identity

$$T_{\mu;v}^v = 0. \tag{14}$$

This gives, for the spherically symmetric case,

$$T_2^2 = T_3^3 = -\frac{R}{2}\Gamma_{10}^0 T_0^0, \tag{15}$$

where $\Gamma_{10}^0 = A'/2A$ is discontinuous in the limit. In this case, however, we can calculate it by following through the steps using Einstein equations. Thus, we write

$$T_2^2 = T_3^3 = \frac{1}{32\pi G}(B-1)\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left[R\left(\frac{1}{B}-1\right)\right]. \tag{16}$$

Simple manipulation then yields (13). Thus, in this case, we have a discontinuous solution which obeys Einstein equation in the sense of distributions by virtue of being the limit of a sequence of continuous solutions.

3. Model with discontinuous "A"

The model given above tells us that discontinuous metrics can be reasonable. We, therefore, look for a model with discontinuous A. It will be seen from eq. (7) that discontinuity in A will lead to a δ-function in T_1^1 and to the gradient of a δ-function in $T_2^2 = T_3^3$. The only possible exception is when $1/B = 0$ at precisely the point of discontinuity of A. It is also necessary that $1/B$ should be continuous at this point. We note from eq. (6) that while $1/B$ is related to T^0_0 by a first order differential equation, the relationship of A to T^0_0 is somewhat more complicated and non-linear except in the weak field limit when $A = 1 + 2\Phi$ where Φ is the gravitational potential. Then a discontinuity in A can arise from a mass dipole layer which, however, does not exist. While we do not yet know the general conditions under which A can become discontinuous, we shall give an interpolating solution of Einstein equations for which, in the limit, A does become discontinuous on a spherical surface where $1/B$ vanishes. (If the solution were relevant to a black hole, this surface would be the event horizon).

We begin by writing (7) as

$$\frac{A'}{A} = -\frac{8\pi GRT^1_1 + (1/R)((1/B) - 1)}{1/B} \equiv \frac{C(R)}{1/B}. \tag{17}$$

The model that we seek is obtained by joining Einstein model interior to a Schwarzschild exterior, so the limiting values of A and B are fixed:

$$A \to \pm\Theta(R_g - R) + \left(1 - \frac{R_g}{R}\right)\Theta(R - R_g), \tag{18}$$

$$(1/B) \to \left(1 - \frac{R^2}{R_g^2}\right)\Theta(R_g - R) + \left(1 - \frac{R_g}{R}\right)\Theta(R - R_g), \tag{19}$$

$$T^0_0 \to \rho_1\Theta(R_g - R) + \ldots, \tag{20}$$

$$T^1_1 \to P_1\Theta(R_g - R) + \ldots, \tag{21}$$

$$T^2_2 = T^3_3 \to P_1\Theta(R_g - R) + \ldots, \tag{22}$$

$$\rho_1 = 3P_1 = \frac{3}{8\pi GR_g^2}, \tag{23}$$

(Θ denotes the usual step function). In eq. (18) the plus sign corresponds to the usual Einstein model while minus sign gives the Euclidean Einstein model. In eqs. (20–22) the dots correspond to possible surface singularities (terms proportional to $\delta(R_g - R)$). The quantities ρ_1, P_1 include both the matter part as well as the contribution in the static limit of a cosmological constant. This determines the model, its physical motivation has been given in [4] and will not concern us here. One can check easily that Einstein equations are satisfied everywhere except at $R = R_g$ where $R_g = 2GM$ as before. This quantity will be kept fixed.

We shall now try to define a sequence of sufficiently smooth metrics which depend on a parameter l and reduce to eqs. (18 & 19) as $l \to 0$. To this end we introduce a C^∞ function $f(R)$ depending on l with $f(R) \to \Theta(R_g - R)$ as $l \to 0$ (in the sense of distributions). The function $f(R)$ will be further specified below. We choose

$$T^0_0 = \rho_1 f(R). \tag{24}$$

(In these models T^0_0 does not have a singular part). We then get

$$\frac{1}{B} = 1 - \frac{3}{R_g^2 R}\int_0^R x^2 f(x)\,\mathrm{d}x. \tag{25}$$

We look for models in which for some R, say $R = R_0$,

$$\frac{1}{B}(R_0) = \left(\frac{1}{B}\right)'(R_0) = 0, \tag{26}$$

$$\left(\frac{1}{B}\right)''(R_0) \sim \frac{1}{lR_g} > 0. \tag{27}$$

(R_0 will be close to and in the limit equal to R_g). Here l is of the order of the thickness of the transition layer where $f(R)$ changes from ≈ 1 to ≈ 0. Equation (25) shows that, for $f(R)$ relevant to the present problem, $1/B$ will have a minimum for some R. If this minimum value of $1/B$ is strictly positive, calculations show that the desired discontinuities do not develop in A in the limit as $l \to 0$. We shall, therefore, assume that $1/B$ vanishes at its minimum, thus getting (26–27). We shall not consider the case when $1/B$ becomes negative. It is also convenient to fix the functions $f(R)$ for any l by requiring

$$\int_0^\infty x^2 f(x)\,\mathrm{d}x = \frac{R_g^3}{3}. \tag{28}$$

It will be shown in Appendix A that such a class of functions satisfying (26) and (28) exists. Going back to (17) we now see that the RHS of that equation has a pole at $R = R_0$ so A can be determined only when we give some prescription to handle that pole. Three common prescriptions are obtained on using

$$\frac{1}{(1/B) \pm i\eta} \quad \text{and} \quad \frac{1}{2}\left(\frac{1}{(1/B) + i\eta} + \frac{1}{(1/B) - i\eta}\right)$$

with $\eta \to 0$ being understood. The first two correspond to using infinitesimal semicircle contours around $R = R_0$ either in the upper half or in the lower half plane; the last one defines principal value prescription. For either prescription to make sense it is necessary that

$$C(R_0) = 0. \tag{29}$$

We, therefore, have

$$\ln A = -\int_R^\infty \frac{C(R')}{1/B(R')}\,\mathrm{d}R', \tag{30}$$

If, for $R < R_0$, we use the principal value prescription, we get our first solution A_1. If we use either of the other prescriptions, we get

$$A_2 = \varepsilon(R - R_0) A_1(R), \tag{31}$$

where

$$\varepsilon(R - R_0) = \begin{cases} +1 & \text{if} \quad R > R_0 \\ -1 & \text{if} \quad R < R_0 \end{cases}.$$

We shall further restrict $C(R)$ so that $A_1 \to +1$ and $A_2 \to -1$ in the deep interior ($R_g - R \gg 1$). This is achieved by requiring

$$P\int_0^\infty \frac{C(R')}{1/B(R')}\,\mathrm{d}R' = 0, \tag{32}$$

(P denotes principal value prescription at $R = R_0$), and we require for the validity of eq. (31)

$$C'(R_0) = \frac{1}{2}\left(\frac{1}{B}\right)''(R_0). \tag{33}$$

A general form of $C(R)$ satisfying these conditions is

$$C(R) = \frac{1}{2}\left(\frac{1}{B}\right)' + \varphi'(R)\frac{1}{B}, \tag{34}$$

where $\varphi(R)$ is any continuously differentiable function with $\varphi(\infty) = \varphi(0) = 0$. It is, furthermore, necessary to choose $\varphi(R)$ in such a way that, in the limit $(l \to 0), A = 1/B$ for $R > R_g$ and $A' = 0$ for $R < R_g$. This is satisfied if we take

$$\varphi(R) = F(R)\ln\left(\frac{1}{B}\right), \tag{35}$$

where $F(R) \approx -\frac{1}{2}$ for $R \ll R_g$ and $\approx \frac{1}{2}$ for $R \gg R_g$ and is smooth but becomes $\frac{1}{2}\varepsilon(R - R_g)$ in the limit $l \to 0$.

As one choice, we can take

$$F(R) = \frac{1}{2}\left(1 - 2f(R) - \frac{f(R)(1 - f(R))}{f(R_0)(1 - f(R_0))} \times (1 - 2f(R_0))\right) \tag{36}$$

(We also demand $F(R_0) = 0$ for definiteness). Recalling that $f(R) \approx 1$ if $(R - R_0)/l \ll 0$ and ≈ 0 if $(R - R_0)/l \gg 0$, we see that $F(R) \approx -\frac{1}{2}$ if $(R - R_0)/l \ll 0$ and $\approx +\frac{1}{2}$ if $(R - R_0)/l \gg 0$. Actually $f(R)$ and $F(R)$ change from their interior to exterior values in a distance of the order of l. Then, integration gives

$$A_1 = \left(\frac{1}{B}\right)^{(1/2)+F(R)} \equiv \exp\left\{\left(\frac{1}{2} + F(R)\right)\ln\left(\frac{1}{B}\right)\right\}, \tag{37}$$

while

$$A_2 = \varepsilon(R - R_0)(1/B)^{(1/2)+F(R)}, \tag{38}$$

We note that A_1 is continuous with a discontinuous first derivative but A_2 has only a mildly singular (with logarithmic singularity) second derivative at $R = R_0$.

Let us now use these interpolated metrics to calculate the singular parts of T_μ^v. Because $1/B$ stays continuous in the limit $l \to 0, T_0^0$ does not gain any δ-function singularity. We shall now show that A'/AB (and hence T_1^1) has no δ-function singularity either. We see that for either A_1 or A_2,

$$\frac{A'}{AB} = \left(\frac{1}{2} + F(R)\right)\left(\frac{1}{B}\right)' + F'(R)\frac{1}{B}\ln\frac{1}{B}. \tag{38}$$

The first term tends to a discontinuous limit which is nonsingular. Any singular contribution can come from the second term only because $F'(R) \sim 1/l$ in the region $|R - R_g| \sim l$. In this region $1/B$ is $O(l)$. Thus

$$\int \mathrm{d}RF'(R)\frac{1}{B}\ln\frac{1}{B} \sim O(l\ln l) \to 0, \tag{39}$$

as $l \to 0$. (The range of integration also contributes a factor l.) Thus A'/AB is not singular.

Singular contributions to $G_2^2 = G_3^3$ in the limit $l \to 0$ come from

$$G_2^2 = G_3^3 = \frac{d}{dR}\left[\frac{A'}{2AB}\right] + \frac{A'^2}{4A^2B} - \frac{A'}{4A}\left(\frac{1}{B}\right)' + \text{nonsingular terms}. \tag{40}$$

The first term is easily calculated using for a fixed $\psi(R) \in C_0^\infty(0,\infty)$

$$\int_0^\infty dR\psi(R)\frac{d}{dR}\left[\frac{A'}{2AB}\right] = -\int_0^\infty dR\psi'(R)\frac{A'}{2AB}. \tag{41}$$

Substituting eq. (38) and taking the limit $l \to 0$ we see that the contribution of the second term vanishes while the first gives

$$\frac{1}{2}\left(\frac{1}{2} + F(R)\right)\left(\frac{1}{B}\right)' \to \frac{R_g}{2R^2}\Theta(R - R_g). \tag{42}$$

This helps to isolate the singular part as

$$\frac{1}{2R_g}\delta(R - R_g).$$

This part is determined by long range effects i.e. it does not depend on the precise form of the interpolating function f or F. The other two terms can be combined and their singular part isolated. The final result is

$$G_2^2 = G_3^3 = \frac{1}{2R_g}\delta(R - R_g) + (I_1 + I_2)\delta(R - R_g) + \text{nonsingular terms}, \tag{43}$$

where

$$I_1 = \frac{1}{4}\int_0^\infty dRF'(R)^2\frac{1}{B}\left(\ln\frac{1}{B}\right)^2, \tag{44}$$

$$I_2 = \frac{1}{4}\int_0^\infty dR\left(\frac{1}{4} - F^2(R)\right)\left(\frac{1}{B}\right)''\ln\frac{1}{B}, \tag{45}$$

where $l \to 0$ is implied. These expressions are the same whether A_1 or A_2 is used. For the difference between A_1'/A_1 and A_2'/A_2 depends on the prescription for handling the pole at $R = R_0$ in eq. (17) but there are no poles left after multiplication by $1/B$. To write eqs (44–45) in an explicit form after taking the indicated limits, we recall that the functions $f(R)$ and $F(R)$ change over from their interior to exterior values in a distance of the order of l around $\bar{R}_g$ which is close to R_g. (The two differ by quantities which are at most of order l). Thus we write generally (for arbitrary functions Φ, χ)

$$f(R) = \Phi\left(\frac{R - \bar{R}_g}{l}\right), \tag{46}$$

$$F(R) = \chi\left(\frac{R - \bar{R}_g}{l}\right). \tag{47}$$

In writing eq. (47) we are allowing χ to be somewhat more general than eq. (36) while satisfying the same constraints. Using these functional forms and carrying out the indicated limits (see Appendix A) we get

$$I_1 = \frac{1}{4R_g}\int_{-\infty}^{\infty} \mathrm{d}y\chi'(y)^2\left[y-y_0-3\int_{y_0}^{y}\mathrm{d}x\,\Phi(x)\right]$$
$$\times\left\{\ln(y-y_0-3\int_{y_0}^{y}\mathrm{d}x\,\Phi(x))-\ln\frac{R_g}{l}\right\}^2 \tag{48}$$
$$I_2 = \frac{3}{4R_g}\int_{-\infty}^{\infty}\mathrm{d}y\left(\frac{1}{4}-\chi^2(y)\right)\Phi'(y)\left\{\ln\frac{R_g}{l}-\ln(y-y_0-3\int_{y_0}^{y}\mathrm{d}x\,\Phi(x))\right\}, \tag{49}$$

where $R_0 = \bar{R}_g + ly_0$. In writing these equations all terms which vanish with l have been dropped. Note that I_1 and I_2 and, hence, the surface tension in $T_2^2 = T_3^3$ is singular in the limit as $l \to 0$ with a logarithmic singularity. We see explicitly that I_1 and I_2 are determined by short range effects. We see this in the dependence of I_1 and I_2 on the functional form of χ and Φ.

4. Conclusions

In this paper, we have considered the circumstances in which a static, spherically symmetric solution of Einstein equation can develop discontinuities in the metric as well as the sense in which these discontinuous metrics obey Einstein equations with an energy-momentum tensor which has δ-function singularities on the surface of discontinuity. Since a static spherically symmetric metric has essentially only two nontrivial components (A, B above), we have discussed two models. A model in which B is discontinuous is discussed first. The metric of this model is obtained as the limit of a sequence of smooth metrics and the limit obeys Einstein equations in the sense of distributions with energy-momentum tensors which are well defined as δ-functions. In particular, these δ-functions are the same whatever functions are used to construct the interpolating sequences. Physically, this means that the form of the limiting solution and the corresponding energy-momentum tensor are determined entirely by long range effects and are not sensitive to short distance effects which could possibly vitiate the validity of Einstein equations.

The other solution is more interesting. We restrict ourselves from the beginning to the case when the energy-momentum tensor does not develop singularities corresponding to the gradient of a δ-function. We do this on grounds of simplicity and physical reasons (e.g. nonexistence of mass dipole layer). It is, then, easy to show that, under these conditions, a discontinuous A is possible only if $1/B$ remains continuous and vanishes on the surface of discontinuity of A. We construct a sequence of smooth metrics whose limit develops the desired discontinuity. (We give two possible sequences, A_2 has a continuous first derivative while A_1 is only continuous). When these sequences are used in the Einstein equations to calculate the corresponding energy-momentum tensors and a limit is taken thereafter, it is found that a surface tension (which manifests itself in a δ function singularity on the surface of discontinuity) is needed. This surface tension is logarithmically singular with a coefficient which depends on the precise functions used to construct interpolating sequences. In this sense it differs from distributions. Physically, this means that the surface tension is sensitive to short distance (or equivalently, from uncertainty principle, high energy) effects. To see this, note that the functions like Φ and

χ determine the way in which the various quantities (metric, components of energy-momentum tensor etc.) change from their interior values to their exterior values in distances of the order of l. At a fundamental level, in a theory involving gravitation, l can only be of the order of Planck length. Sensitivity to details on such a length scale clearly implies sensitivity to high energy processes. It is also interesting to note that the surface tension is the same whether we use A_1 or A_2. Thus the important thing is the discontinuity in A. The surface tension is not sensitive to the change in the signature of the metric in the case when $A \rightarrow -1$.

We see then that the limiting solution with discontinuous A does not quite satisfy Einstein equations (in the sense of distributions) and will have to be rejected if Einstein equations are strictly valid. But general relativity (and Einstein equations) is only the low energy limit of a (as yet unknown) more fundamental theory which includes quantum effects and considers even Planck level processes. In such a theory phenomena analogous to discontinuous metrics may or may not occur but it is only in the context of such a quantum theory that we can calculate the associated surface tension if they (metrics with discontinuous A) exist and are relevant to physics (e.g. black holes). The basic reason for this apparent breakdown of Einstein equations here is the nonlinearity of the equations. In linear theories, limits of sequences of solutions always satisfy the relevant equation in the sense of distributions.

In conclusion, we have shown that metrics with discontinuities can be respectable both when these are due to long range processes and when they are due to short range processes. In both cases, metrical discontinuity is related to the existence of a surface tension. It should be noted that while the role of surface tension in producing pressure difference is well known in classical physics, the relationship of the former to metrical discontinuity is a new result as far as we know. Two other relevant points must be made in this connection. If it is desired that the metric be real and have, at least, a continuous first derivative for all finite $l(\neq 0)$ then one must choose one of the semicircular contours in eq. (30) to handle the pole at $R = R_0$. The second point concerns the acceptability of infinite surface tension. Normally it is not acceptable. For example, we cannot take the limit $R_0 \rightarrow R_g + 0$ in eq. (13). The infinity in this case arises from long distance effects for which general relativity is adequate. On the other hand, the infinity that arises in the second model is an ultraviolet infinity which means that general relativity is not adequate for this problem. This point must be made to rule out arbitrary joining of metrics.

Appendix A

The basic requirement on $f(R)$ is that it changes from ≈ 1 to ≈ 0 in a distance of the order of l. A trial solution is of the form

$$f(R) = f_0(R) + \alpha f_0(R)(1 - f_0(R)), \tag{A1}$$

where

$$f_0(R) = \left[1 + \exp\left(\frac{R - \bar{R}_g}{l}\right)\right]^{-1}, \tag{A2}$$

and α is an unknown number while $\bar{R}_g$ will be determined below. For this we use eq. (28). Straightforward calculation gives for small l,

$$\bar{R}_g = R_g - \alpha l + \frac{l^2}{R_g}\left(\alpha^2 - \frac{\pi^2}{3}\right). \tag{A3}$$

We now use eq. (26). Writing $R_0 = \bar{R}_g + l x_0$, we find that x_0 and α satisfy the following equations as $l \to 0$

$$\int_{x_0}^{\infty} \frac{\mathrm{d}x}{1+e^x} + \frac{\alpha}{1+e^{x_0}} = \frac{1}{3}(\alpha - x_0), \tag{A4}$$

$$\frac{1}{1+e^{x_0}} + \frac{\alpha}{(1+e^{x_0})(1+e^{-x_0})} = \frac{1}{3}. \tag{A5}$$

These equations have two solutions. They are approximately

$$x_0 = 2.468, \quad \alpha = 3.54,$$

and

$$x_0 = -1.20, \quad \alpha = -2.54.$$

This shows that functions $f(R)$ with indicated properties exist.

We now indicate the steps leading to the expressions for I_1 and I_2 in eq. (48–49). We start from the second and third term in eq. (40) and use (38) to get

$$\frac{A'^2}{4A^2B} - \frac{A'}{4A}\left(\frac{1}{B}\right)' = \frac{1}{4}\left[\left(F^2(R) - \frac{1}{4}\right)\frac{(1/B)'^2}{1/B} + \left\{\frac{\mathrm{d}}{\mathrm{d}R}\left(F^2(R) - \frac{1}{4}\right)\right\} \times \left(\frac{1}{B}\right)' \ln\frac{1}{B} + F'^2(R)\left(\frac{1}{B}\right)\left(\ln\frac{1}{B}\right)^2\right]. \tag{A6}$$

Given a function $\Psi(R) \in C_0^{\infty}(0,\infty)$ as before, we can see that

$$\int_0^{\infty} \mathrm{d}R\Psi(R)\left\{\frac{\mathrm{d}}{\mathrm{d}R}\left(F^2(R) - \frac{1}{4}\right)\right\}\left(\frac{1}{B}\right)' \ln\frac{1}{B} = \int_0^{\infty} \mathrm{d}R\left(\frac{1}{4} - F^2(R)\right) \times \left\{\Psi'(R)\left(\frac{1}{B}\right)' \ln\frac{1}{B} + \Psi(R)\left(\frac{1}{B}\right)'' \ln\frac{1}{B} + \Psi(R)\frac{(1/B)'^2}{1/B}\right\}. \tag{A7}$$

The first term does not contribute as $l \to 0$ vanishing as $O(l \ln l)$. The third term cancels the first term in (A6). The remaining terms are recognized as δ-functions. (Recall that $\frac{1}{4} - F^2(R)$ and $F'(R)$ are both sharply peaked). Thus we get (44–45). We now write

$$\frac{1}{B} = -\frac{3}{R_g^2 R_0}\int_{R_0}^{R} x^2 f(x)\,\mathrm{d}x + \frac{3}{R_g^2}\left(\frac{1}{R_0} - \frac{1}{R}\right)\int_0^{R} x^2 f(x)\,\mathrm{d}x. \tag{A8}$$

(This follows from (25) on using (26).) Similarly we get

$$\left(\frac{1}{B}\right)'' = -\frac{6}{R_g^2 R^3}\int_0^{R} x^2 f(x)\,\mathrm{d}x - \frac{3R}{R_g^2} f'(R). \tag{A9}$$

Substituting in (45) and making the change of variable to $y = R - \bar{R}_g/l$ with $y_0 = R_0 - \bar{R}_g/l$, we easily derive (49). Equation (48) follows similarly.

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