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# Spherically symmetric static solutions of the Einstein–Maxwell equations

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**Abstract.** We report a new formalism to obtain solutions of Einstein–Maxwell's equations for static spheres assuming the matter content to be a charged perfect fluid of null-conductivity. Defining three new variables $u = 4\pi\mathcal{E}r^2, v = 4\pi pr^2$ and $w = (4\pi/3)(\rho + \varepsilon)r^2$ where $\mathcal{E}, \rho$ and $\varepsilon$ denote respectively energy densities of the electric, matter and free gravitational fields whereas $p$ is the fluid pressure, Einstein's field equations are rewritten in an elegant form. The solutions given by Bonnor [1], Nduka [2], Cooperstock and De la Cruz [3], Mehra [4], Tikekar [5, 6], Xingxiang [7], Patino and Rago [8] are all shown to possess simple relations between $u, v$, and $w$ whereas Pant and Sah's [9] solution for which all the three functions, $u, v$, and $w$ are constants is a trivial case of the present formalism, We have presented six new solutions with $\varepsilon = 2\rho$. For the first three solutions $w$ and $u$ are constants with $v$ as a variable whereas the remaining three solutions satisfy the equation of state for isothermal gas; $v = kw = -ku$ where (i) $k$ is an arbitrary constant but not equal to 1 or 1/3 (ii) $k = 1$ and (iii) $k = 1/3$. We also obtained a generalization of Cooperstock and De la Cruz's [3] solution which is regular for $2\rho > \varepsilon$ but singular for $2\rho \leq \varepsilon$.

**Keywords.** Einstein–Maxwell's equations; energy-momentum tensor; energy density of matter; energy density of the free gravitational field; singularities.

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## 1. Introduction

In general relativity static charged fluid spheres have been discussed by many authors including Bonnor [1], Nduka [2], Cooperstock and De la Cruz [3], Mehra [4], Tikekar [5, 6], Xingxiang [7], Patino and Rago [8], and Pant and Sah [9]. These solutions have been adopted as the sources for the well-known Reissner–Nordstrom solution which describes the gravitational field exterior to such charged spheres. From the analysis of the Reissner–Nordstrom solution Graves and Brill [10] have shown that it has an oscillatory character. On an initial surface this spacetime shows the same general behaviour as the Schwarzschild exterior spacetime describing a wormhole or bridge between two asymptotically flat-spaces but with an electric flux flowing through the wormhole. The pressure of the electric flux acts as a cushion which might prevent a collapsing object provided a neat balance is maintained between the mass and charge of the body. Bonnor [1] and independently De and Raychaudhuri [11] have shown that in the case of a static distribution of pressureless charged fluid (dust) the matter density $\rho$ is equal to the absolute value of the charge density $|\sigma|$. In the present paper we develop a new formalism to study static charged perfect fluid filled spheres which will help us to analyse the role of the energy density of the free gravitational field in relation to matter and charge densities.

It is well-known that the Riemann curvature tensor can be uniquely decomposed into the conformal Weyl curvature tensor, the Ricci tensor and its scalar. Thus, the total curvature of the spacetime as described by the Riemann curvature tensor comes from the contribution of the free gravitational field as represented by the conformal Weyl tensor and matter/charge distribution as described by the Ricci tensor and its scalar curvature. When the material distribution consists of a charged perfect fluid of proper material density $\rho$, pressure $p$ and electrostatic energy $\mathcal{E}$ one can compare these physical variables with the eigenvalue of the conformal Weyl tensor $\varepsilon$ to test the dominating features of a particular solution. In the present paper we have given some new solutions of Einstein–Maxwell's equations for spherically symmetric static spacetimes with $\varepsilon = 2\rho$. The metric form and field equations are given in § 2. By introducing new variables $u, v, w$ in place of $\mathcal{E}, p, (\rho+\varepsilon)$ respectively, the field equations are cast into suitable form in § 3. In § 4 we derive new solutions of the field equations with certain assumptions. In § 5 we give a generalization of Cooperstock and De la Cruz's [3] solution and the conclusion is given in § 6.

## 2. The metric and the field equations

The metric for a static spherically symmetric spacetime may be written in the form

$$ds^2 = -\exp\lambda\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \exp\nu\, dt^2, \tag{1}$$

where $\lambda$ and $\nu$ are functions of the radial coordinate $r$ only.

Assuming that the spacetime described by (1) is filled with a charged perfect fluid, the energy momentum tensor is written as

$$T_a^b = (\rho + p)u_a u^b - p g_a^b - F_{ac}F^{bc} + (1/4)g_a^b F_{cd}F^{cd}, \tag{2}$$

where $u_a u^a = 1$. In view of the spherical symmetry and static nature of the spacetime we get $u^a = (0, 0, 0\exp(-\nu/2))$ and the only non-vanishing component of the skew-symmetric Maxwell tensor $F_{ab}$ is given by $F_{14}$. We assume that the charged fluid is of null conductivity so that the 4-dimensional charge-current vector is written as $J^a = \sigma u^a$ where $\sigma$ denotes the density of electric charge. From the Maxwell equations

$$F_{ab,c} + F_{bc,a} + F_{ca,b} = 0, \tag{3}$$

$$\{(-g)^{1/2}F^{ab}\}_{,b} = (-g)^{1/2}J^a, \tag{4}$$

we get

$$F_{14} = -\exp\{(\lambda+\nu)/2\}Q(r)/r^2, \tag{5}$$

$$J_4 = -\exp\{(\nu-\lambda)/2\}Q'(r)/r^2, \tag{6}$$

where $Q(r)$ is the total charge contained in a sphere of coordinate radius $r$ (Nduka [2]) and a prime denotes a differentiation with respect to $r$.

With the help of Einstein's field equations we connect (1) and (2) as below:

$$8\pi(p - \mathcal{E}) = -r^{-2}\{1 - \exp(-\lambda)\} + r^{-1}\exp(-\lambda)\nu', \tag{7}$$

$$8\pi(p + \mathcal{E}) = -8\pi\varepsilon + r^{-2}\{1 - \exp(-\lambda)\} + r^{-1}\exp(-\lambda)(\nu' - \lambda'), \tag{8}$$

$$8\pi(\rho + \mathcal{E}) = r^{-2}\{1 - \exp(-\lambda)\} + r^{-1}\exp(-\lambda)\lambda', \tag{9}$$

where a prime for $\lambda$ and $\nu$ denotes a differentiation with respect to the radial coordinate $r$. In classical electromagnetic theory the vacuum field energy density is defined by $\mathcal{E} = (\bar{E}^2 + \bar{H}^2)/2$ which in the present case reduces to $\mathcal{E} = (\bar{E}^2/2) = (-1/2)F_{14}F^{14}$. In (8) $\varepsilon$ on the right-hand side represents the eigenvalue of the conformal Weyl curvature tensor [12] denoting the energy density of the free gravitational field [13] and is given by

$$8\pi\varepsilon = r^{-2}\{1 - \exp(-\lambda)\} - (1/4)\exp(-\lambda)\{2\nu'' + (\nu' - \lambda')(\nu' - 2r^{-1})\}. \tag{10}$$

Now, making the combination of $\{(8) + (9) - (7)\}$, we get

$$\exp(-\lambda) = 1 - (8\pi/3)(\rho + \varepsilon + 3\mathcal{E})r^2. \tag{11}$$

It may be noted that the energy densities of matter, free gravitational and electrostatic fields are coupled and for simplicity the coupling constants are chosen as unity. These relative densities will play a decisive role in determining the strength of a singularity.

## 3. Alternative form of the field equations

The Einstein field equations given in § 2 are rather vague. To have a clear understanding of these equations we introduce three physical variables $u, v, w$ respectively in place of $\mathcal{E}$, $p$, $(\rho + \varepsilon)$ as below:

$$u = 4\pi\mathcal{E}r^2, \tag{12}$$

$$v = 4\pi p r^2, \tag{13}$$

$$w = (4\pi/3)(\rho + \varepsilon)r^2. \tag{14}$$

Hence, using (12) and (14) in (11), we get

$$\exp(-\lambda) = 1 - 2(w + u). \tag{15}$$

We express (7) by using (12), (13) and (15) as

$$r\nu' = \frac{2(v + w)}{1 - 2(w + u)}. \tag{16}$$

Again using the definitions (12), (13), (14) in (7), (8), (9) and making the combination of $\{(7) - (8) + 2 \times (9)\}$ we get,

$$\frac{(w' + u')}{r} = \frac{4\pi}{3}(2\rho - \varepsilon). \tag{17}$$

Thus, when $2\rho = \varepsilon$ we get $w + u =$ constant, [or $\exp(\lambda) =$ constant] and conversely. Such models are particularly simple because $\rho, \varepsilon$ and $\mathcal{E}$ fall off as $1/r^2$, when $w$ and $u$ are constants.

The radial equation of motion, (i.e. equation of hydrostatic equilibrium) with the help of (5) and (6) takes the form

$$(\rho + p)\nu' = -2p' + \frac{(Q^2)'}{r^4}. \tag{18}$$

We recast (18) after eliminating $\nu'$ with the help of (16) into the form

$$\frac{\mathrm{d}r}{r} = \frac{(v+w)(\mathrm{d}u + \mathrm{d}w) + a(u+w)(\mathrm{d}v\,\mathrm{d}u)}{H} \tag{19}$$

where

$$H \equiv \{2(u+v) - 4(u+w)(u+v) - (v+w)^2\}. \tag{20}$$

Finally, eliminating $\lambda'$ from (9) with the help of (15) and then using (19) we get,

$$4\pi(\rho + \mathcal{E})r^2 = (u+w) + \frac{H(\mathrm{d}u + \mathrm{d}w)}{\{1-2(u+w)\}(\mathrm{d}v - \mathrm{d}u) + (v+w)(\mathrm{d}u + \mathrm{d}w)}. \tag{21}$$

Thus, we have essentially three equations (17), (19) and (21) to determine five unknown quantities $u, v, w, \lambda$ and $\nu$. Hence, we are at liberty to impose two conditions on these five unknown quantities. In case these two relations are between $u, v,$ and $w$ the above formulation will be very helpful.

The functional relationship between $u, v,$ and $w$ for the solutions mentioned earlier can be computed very easily and in some cases the relationship is linear. We describe in the next section two cases in which (I) $u$ and $w$ are constants but $v$ is a variable and (II) $u, v, w$ are all variables.

## 4. Solutions of the field equations

In this section we derive new solutions of the field equations given in § 3. The starting point for this derivation is eq. (17). Thus, assuming

$$\varepsilon = 2\rho \tag{22}$$

we get the following two cases:

Case I: $u$ and $w$ being constants so that

$$u = 4\pi\mathcal{E}r^2 = \text{constant} = 2\pi b^2 \quad \text{(say)}, \tag{23}$$

$$w = (4\pi/3)(\rho + \varepsilon)r^2 = 4\pi\rho r^2 = \text{constant}. \tag{24}$$

Case II: $u$ and $w$ are variables and for simplicity we choose

$$u + w = 0. \tag{25}$$

We first consider Case I and using (22) and (23) in (19) we get

$$-\frac{\mathrm{dr}}{\mathrm{r}} = \frac{\alpha\,\mathrm{d}v}{(v-\beta)^2 + (\gamma - \beta^2)}, \tag{26}$$

where the constants $\alpha, \beta, \gamma$ are defined as

$$\alpha = 1 - 2(u + w), \tag{27}$$

$$\beta = \alpha - w, \tag{28}$$

$$\gamma = (2u + w)^2 - 2u. \tag{29}$$

We note from (26) that Pant and Sah's [9] solution for which besides $u$ and $w, v$ is also a constant is a trivial solution of the present formalism. We shall integrate (26) assuming the following:

$$\text{(i)} \quad \beta \neq 0, |\gamma - \beta^2| > 0, \tag{30}$$

$$\text{(ii)} \quad \beta \neq 0, \beta^2 - \gamma = 0. \tag{31}$$

Thus, integrating (26) we get

$$v = \beta + \begin{cases} b_1 \tan\left\{\dfrac{b_1}{\alpha} \log\left(\dfrac{a}{r}\right)\right\}; & w > \dfrac{1}{4}, \\ b_2\left(\dfrac{r^{2b_2/\alpha} - d_2}{r^{2b_2/\alpha} + d_2}\right); & w < \dfrac{1}{4}, \\ \dfrac{\alpha}{\log(r/\delta)}; & w = \dfrac{1}{4}, \end{cases} \tag{32}$$

where $b_1, b_2, d_2$ and $\delta$ are constants.

Using (25), (28) and (32) in (16), we get

$$\exp \nu = \begin{cases} k_1 r^2 \left\{1 + \tan\left(\dfrac{b_1}{\alpha} \log a\right) \tan\left(\dfrac{b_1}{\alpha} \log r\right)\right\} \cos^2\left(\dfrac{b_1}{\alpha} \log r\right); & w > \dfrac{1}{4}, \\ k_2 r^2 \{r^{(b_2/\alpha)} + d_2 r^{(-b_2/\alpha)}\}^2; & w < \dfrac{1}{4}, \\ d^2 r^2 \{\log(r/\delta)\}^2; & w = \dfrac{1}{4}, \end{cases} \tag{33}$$

$k_1, k_2$ and $d$ being constants.

Using (15) and (32) in (5) and with the help of (22), we get

$$F_{14} = \begin{cases} -b(k_1/\alpha)^{1/2} \left\{1 + \tan\left(\dfrac{b_1}{\alpha} \log \alpha\right) \tan\left(\dfrac{b_1}{\alpha} \log r\right)\right\}^{1/2} \\ \quad \times \cos\left(\dfrac{b_1}{\alpha} \log r\right); \quad w > \dfrac{1}{4}, \\ -b(k_2/\alpha)^{1/2} \{r^{(b_2/\alpha)} + d_2 r^{(-b_2/\alpha)}\}; \quad w < \dfrac{1}{4}, \\ -(\alpha)^{-1/2} b d \log(r/\delta); \quad w = \dfrac{1}{4}, \end{cases} \tag{34}$$

For all the three cases

$$J_{(4)} = \sigma = \frac{\alpha^{1/2} F_{(14)}}{r} = \frac{-b\alpha^{1/2}}{r^2}, \tag{35}$$

$$Q(r) = br, \tag{36}$$

showing that the total charge contained within a sphere is proportional to the radius of the sphere itself. In (35) $J_{(4)}$ and $F_{(14)}$ represent the physical components of the respective quantities.

Since $v = 0$ at $r = r_b$, from (32) we can obtain the expressions for $r_b$ and in all the three cases we have verified that these interior solutions can be matched with the exterior Reissner–Nordstron solution by choosing the arbitrary constants appropriately.

Now, we consider Case II where $u + w = 0$ (or $\rho + \mathcal{E} = 0$) with $u$ and $w$ being variables. We further assume the pressure-density relation for isothermal gas spheres as

$$v = kw(= -ku). \tag{37}$$

Thus, substituting (25) and (37) in (19) and then integrating we get

$$u = \frac{2\eta}{\mu^2 - \gamma r^{2\eta/\mu}}, \tag{38}$$

where $\mu = 1 + k, \eta = 1 - k$ and $\gamma$ is a constant of integration. Hence $v$ and $w$ are known through (37). Using (37) and (38) in (16) we get the expression for $\nu$ as

$$\exp \nu = \left(1 - \frac{\mu^2}{\gamma r^{2\eta/\mu}}\right)^2. \tag{39}$$

It is easy to calculate the expressions for $F_{(14)}, J_{(4)}$ and $\sigma$ which are stated below:

$$F_{(14)} = -Q(r)/r^2, \tag{40}$$

$$J_{(4)} = \sigma = -Q'(r)/r^2, \tag{41}$$

where

$$Q(r) = (\eta/\gamma\pi)^{1/2} r^{1-(\eta/\mu)} \left(1 - \frac{\mu^2}{\gamma r^{2\eta/\mu}}\right)^{-1/2}. \tag{42}$$

*Particular solution (i):*

By putting $k = 1$ in (37) so that in view of $\varepsilon = 2\rho$ we get the equation of state for stiff matter [14–16]

$$p = \rho. \tag{43}$$

With the above simplifications in (19) we get on integration

$$u = (\log(\delta/r^2))^{-1}, \tag{44}$$

where $\delta$ is a constant of integration.

We integrate (16), using (44), so that

$$\exp \nu = (\log(\delta/r^2))^2. \tag{45}$$

For the present case the expression for $Q(r)$ is given by

$$Q(r) = \frac{r}{(2\pi)^{1/2}} (\log(\delta/r^2))^{-1/2}. \tag{46}$$

so that $F_{(14)}, J_{(4)}$ and $\sigma$ may be obtained using (40) and (41).

*Particular solution (ii):*

By putting $k = 1/3$ in (37) so that the matter distribution is isotopic radiation with charge, we get from (38), (39) and (42) respectively

$$-u = w = 3v = (3/4)(1 - \bar{\gamma}r)^{-1}, \tag{47}$$

$$\exp\nu = (1/\bar{\gamma}^2 r^2)(1 - \bar{\gamma}r)^2, \tag{48}$$

$$Q(r) = (3/8\pi\bar{\gamma})^{1/2} r^{1/2} \left\{1 - \left(\frac{1}{\bar{\gamma}r}\right)\right\}^{-1/2}, \tag{49}$$

and $\bar{\gamma} = (9\gamma/16)$. From (49) using (40) and (41) we can compute $F_{(14)}$ and $J_{(4)}(= \sigma)$ respectively. Since $p = 0$ implies $\rho = 0$ these solutions have no exteriors.

Finally, we may state that in Case II instead of $u + w = 0$ we could as well take $u + w = \kappa$ where $\kappa$ is a constant. In such a case (19) takes the form

$$\frac{\mathrm{d}r}{r} = \frac{(1 - 2\kappa)(\mathrm{d}v - \mathrm{d}u)}{\{2(1 - 2\kappa)(v + u) - (v + w)^2\}}. \tag{50}$$

The pressure-density relations for isothermal gas spheres given by (37) takes the form

$$v = kw = k(\kappa - u). \tag{51}$$

Using (51) in (50), we get

$$-\frac{\mathrm{d}r}{r} = \frac{(1 - 2\kappa)\mu\mathrm{d}U}{\mu^2 U^2 - 2\eta(1 - 2\kappa)U - 2\kappa(1 - 2\kappa)}, \tag{52}$$

where $U \equiv u - \kappa$. We can bring (52) into the form similar to (26) and integrate.

## 5. Generalized Cooperstock–De la Cruz solution

In this section we give a generalization of Cooperstock and De la Cruz's [3] solution as an example of $2\rho - \varepsilon \neq 0$. For this purpose we choose

$$\exp(-\lambda) = 1 - a^2 r^{n+2}, \tag{53}$$

so that when compared to (11) we get

$$(8\pi/3)(\rho + \varepsilon + 3\mathcal{E})r^2 = 2(u + w) = a^2 r^{n+2}. \tag{54}$$

In addition to the $u$, $w$ relation given by (54) we put

$$v = 0 \tag{55}$$

so that (17), (19) and (21) can be easily solved. Thus, we get the following expressions:

$$u = -\frac{a^2}{2} r^{n+2} + \{1 - (1 - a^2 r^{n+2})^{1/2}\}, \tag{56}$$

$$w = a^2 r^{n+2} - \{1 - (1 - a^2 r^{n+2})^{1/2}\}, \tag{57}$$

$$\nu = 2\log\left[\frac{\beta}{r}\left\{\frac{1 - (1 - a^2 r^{n+2})^{1/2}}{a r^{(n+2)/2}}\right\}^{2/(n+2)}\right], \tag{58}$$

$$Q(r) = \frac{1}{(4\pi)^{1/2}}[-a^2 r^{n+4} + 2r^2\{1 - (1 - a^2 r^{n+2})^{1/2}\}]^{1/2}. \tag{59}$$

Hence,

$$F_{(14)} = -\frac{Q(r)}{r^2}, \tag{60}$$

$$J_{(4)} = \sigma = -(1 - a^2 r^{n+2})^{1/2} Q'(r)/r^2. \tag{61}$$

At the boundary $r = r_b$ we have $\lambda(r_b) + \nu(r_b) = 0$ giving

$$\beta = \frac{a^{2/(n+2)} r_b^2 \{1 - a^2 r_b^{n+2}\}^{1/2}}{[1 - (1 - a^2 r_b^{n+2})^{1/2}]^{2/(n+2)}}. \tag{62}$$

Also we give below the explicit expressions for $\rho, \varepsilon, \mathcal{E}$;

$$8\pi\rho = (n+4)a^2 r^n - (2/r^2)\{1 - (1 - a^2 r^{n+2})^{1/2}\}, \tag{63}$$

$$4\pi\varepsilon = 8\pi\rho - \frac{3a^2}{2}(n+2)r^n, \tag{64}$$

$$8\pi\mathcal{E} = -a^2 r^n + (2/r^2)\{1 - (1 - a^2 r^{n+2})^{1/2}\}. \tag{65}$$

We note from the above that for $n = 0$ these expressions reduce to those given by Cooperstock and De la Cruz [3]. Also from (63) and (64) we get

$$\frac{4\pi}{3}(2\rho - \varepsilon) = \frac{(n+2)a^2 r^n}{2}, \tag{66}$$

giving two possibilities;

$$\text{(i)} \quad 2\rho - \varepsilon > 0 \quad \text{for } n > -2, \tag{67}$$

$$\text{(ii)} \quad 2\rho - \varepsilon < 0 \quad \text{for } n < -2. \tag{68}$$

For small $r$ we note from (59), (61), (63), (64) and (65) that

$$Q(r) = 0, \tag{69}$$

$$\mathcal{E} = 0, \tag{70}$$

$$\sigma \longrightarrow \infty, \tag{71}$$

$$8\pi\rho = (n+3)a^2 r^n, \tag{72}$$

$$8\pi\varepsilon = \left(2 - \frac{n}{2}\right)a^2 r^n. \tag{73}$$

To avoid a singularity in the matter and free gravitational fields near the origin we require $n > 0$.

## 6. Conclusion

The present formalism is more suitable to derive new solutions of Einstein–Maxwell's equations for static spheres. The solution presented in §4 represent very strong gravitational fields with $\varepsilon = 2\rho$ and accordingly they also exhibit singularities at the center. The formalism also helps to generalize known solutions through (17) as in the case of Cooperstock–De la Cruz's solution. We may also mention here that in the case of Cooperstock–De la Cruz's solution as well as its generalization $\rho$ and $|\sigma|$ are not equal since there is no stipulation between $g_{44}(= \exp \nu)$ and the scalar potential $\phi$ where $F_{14} = \phi'$. Since we are dealing with strong gravitational fields such a stipulation does not seem to be necessary. We may further add that it is possible to give a direct generalization of the present formalism to the case of a changed anisotropic fluid of null conductivity by taking two variables $v_r = 4\pi p_r r^2, v_\perp = 4\pi p_\perp r^2$ in place of $v$ where $p_r$ and $p_\perp$ denote respectively the fluid pressure along the radial and transverse directions.

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