

## The structure of the quantum mechanical state space and induced superselection rules

JOACHIM KUPSCH

Theoretical Physics Group, Tata Institute of Fundamental Research, Homi Bhabha Road,  
Mumbai 400 005, India

Permanent address: Fachbereich Physik, Univ. Kaiserslautern, D-67653 Kaiserslautern, Germany

Email: kupsch@physik.uni-kl.de

**Abstract.** The role of superselection rules for the derivation of classical probability within quantum mechanics is investigated and examples of superselection rules induced by the environment are discussed.

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### 1. Introduction

One of the puzzles of quantum mechanics is the question, how classical objects can arise in quantum theory. Quantum mechanics is a statistical theory, but its statistics differs on a fundamental level from the statistics of classical objects. The EPR problem and the violation of Bell's inequalities are consequences of this fact [1–3] (English translation of [2] in [4]). These questions are usually discussed as problems of the interpretation of the Schrödinger wave function. Here I would like to emphasize a more abstract approach which relates these problems to the geometrical structure of the quantum mechanical state space, i.e. the total set of pure and mixed states. It is exactly this structure which forbids to assign an objective probability measure to the states.

It is known since a long time that the statistical results of quantum mechanics become consistent with a classical statistics of “facts”, if the superposition principle is reduced to “superselection sectors”, i.e. coherent orthogonal subspaces of the full Hilbert space. The mathematical structure of quantum mechanics and of quantum field theory provides us with only a few “superselection rules”, the most important being the charge superselection rule related to gauge invariance, see e.g. [5,6] and the references given therein. But there are definitively not enough of these superselection rules to understand classical properties in quantum theory. A possible solution of this problem is the emergence of effective superselection rules due to decoherence caused by the interaction with the environment. These investigations – often related to a discussion of the process of measurement – started in the eighties; some early references are [7–9].

In this talk the transition from quantum mechanics to classical physics is investigated in the following restricted sense: How can a classical statistical theory emerge from quantum probability? The emphasis is laid on the principles of induced superselection. Many aspects, like Markov approximation and localization models are totally omitted. For a comprehensive investigation of decoherence see the book [10]. The talk is organized as follows. The structure of the state space is discussed in § 2. Superselection rules are defined in § 3. In § 4 the dynamics of subsystems and the emergence of effective superselection rules are recapitulated. Some exactly solvable models and mathematical formulations of induced superselection rules are discussed in § 5.

## 2. Structure of the state space

We start with a few mathematical notations. If  $\mathcal{H}$  is a separable Hilbert space we use the following spaces of linear operators defined on  $\mathcal{H}$ .

$\mathcal{B}(\mathcal{H})$ : The  $\mathbf{R}$ -linear space of all bounded selfadjoint operators  $A$ . The norm of this space is the operator norm  $\|A\|$ .

$\mathcal{T}(\mathcal{H})$ : The  $\mathbf{R}$ -linear space of all selfadjoint traceclass operators  $A$ . These operators have a pure point spectrum  $\alpha_i \in \mathbf{R}$ ,  $i = 1, 2, \dots$ , with  $\sum_i |\alpha_i| < \infty$ . The natural norm of this space is the trace norm  $\|A\|_1 = \text{tr}\sqrt{A^+A} = \sum_i |\alpha_i|$ . Another norm, used in the following sections, is the Hilbert–Schmidt norm  $\|A\|_2 = \sqrt{\text{tr}A^+A}$ . These norms satisfy the inequalities  $\|A\| \leq \|A\|_2 \leq \|A\|_1$ .

$\mathcal{D}(\mathcal{H})$ : The set of all statistical operators, i.e. positive traceclass operators  $W$  with a normalized trace,  $\text{tr}W = 1$ .

$\mathcal{P}(\mathcal{H})$ : The set of all rank one projection operators  $P^1$ . These sets satisfy the obvious inclusions  $\mathcal{P}(\mathcal{H}) \subset \mathcal{D}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ .

Any state of a quantum system is represented by a statistical operator  $W \in \mathcal{D}(\mathcal{H})$ , the elements of  $\mathcal{P}(\mathcal{H})$  thereby correspond to the pure states. Any observable is represented by an operator  $A \in \mathcal{B}(\mathcal{H})$ . The use of bounded operators is only an apparent restriction, since the whole information of an unbounded observable can be recovered from its spectral resolution involving only bounded operators. The expectation of the observable  $A$  in the state  $W$  is the usual trace  $\text{tr}WA$ , which is always finite for  $W \in \mathcal{D}(\mathcal{H})$  and  $A \in \mathcal{B}(\mathcal{H})$ . Without additional knowledge about the structure of the system we have to assume that the set of all states corresponds exactly to  $\mathcal{D}(\mathcal{H})$ , and the set of all (bounded) observables to  $\mathcal{B}(\mathcal{H})$ . The state space  $\mathcal{D}(\mathcal{H})$  has an essential property: it is a convex set, i.e.  $W_1, W_2 \in \mathcal{D}(\mathcal{H})$  implies  $\lambda_1 W_1 + \lambda_2 W_2 \in \mathcal{D}(\mathcal{H})$  if  $\lambda_{1,2} \geq 0$  and  $\lambda_1 + \lambda_2 = 1$ . Any statistical operator  $W \in \mathcal{D}(\mathcal{H})$  can be decomposed into pure states

$$W = \sum_n \lambda_n P_n^1 \tag{1}$$

with  $P_n^1 \in \mathcal{P}(\mathcal{H})$  and probabilities  $w_n \geq 0$ ,  $\sum_n w_n = 1$ . An explicit example is the spectral decomposition of  $W$ . But there are many other possibilities, and we shall investigate that aspect in more detail. It is exactly this arbitrariness that does not allow a classical interpretation of quantum probability.

Before we continue with the investigation of the quantum mechanical state space we introduce some notations concerning convex sets. A bounded and closed subset  $\mathcal{M}$  of an  $\mathbf{R}$ -linear space is convex if with two points  $x_1, x_2 \in \mathcal{M}$  also the connecting line segment

$\lambda x_1 + (1 - \lambda)x_2, \lambda \in [0, 1]$ , belongs to  $\mathcal{M}$ . Those points of  $\mathcal{M}$  which do not lie on a line connecting two others, i.e. for which  $x = \frac{1}{2}x_1 + \frac{1}{2}x_2$  with  $x_{1,2} \in \mathcal{M}$  is only possible if  $x_1 = x_2 = x$ , are called extremal. The extremal points are always boundary points, but not necessarily all boundary points are extremal. The boundary of  $\mathcal{M}$  will be denoted by  $\partial\mathcal{M}$ , the set of extremal points – the *extremal boundary* – by  $\partial_e\mathcal{M}$ . An important statement about convex sets is: Any point  $x \in \mathcal{M}$  can be represented by an integral over the extremal boundary

$$x = \int_{\partial_e\mathcal{M}} y d\lambda(y) \tag{2}$$

where  $d\lambda(y)$  is a probability measure, i.e. a non-negative measure with  $\int d\lambda(y) = 1$ , concentrated on (the closure of)  $\partial_e\mathcal{M}$ . This representation has been derived in a rather general context by Choquet, see e.g. [11,12]. In the case of point measures the integral (2) yields the sum

$$x = \sum_i \lambda_i y_i \tag{3}$$

with  $y_i \in \partial_e\mathcal{M}$  and probabilities  $0 \leq \lambda_i \leq 1, \sum_i \lambda_i = 1$ . The representations (2) and (3) are only unique if  $\mathcal{M}$  is a simplex. An  $n$ -dimensional simplex  $\mathcal{M}$  is the closed convex set generated by  $n + 1$  points  $y_i, i = 1, \dots, n + 1$ , where the connecting lines  $y_i - y_{n+1}, i = 1, \dots, n$  are linearly independent,  $\mathcal{M} = \{x = \sum_{i=1, \dots, n} \lambda_i y_i | \lambda_i \geq 0, \sum_i \lambda_i = 1\}$ . In two dimensions this is a triangle, in three dimensions a tetrahedron. The extremal boundary of a simplex is the set of its vertices, and the weights  $\lambda_i$  are the uniquely defined barycentric coordinates of the point  $x \in \mathcal{M}$  with respect to the vertices  $y_i$ . Another class of convex sets are balls  $B^n = \{x \in \mathbb{R}^n, |x| \leq 1\}$ . It is easily seen that in this case the extreme boundary coincides with the boundary  $S^{n-1} = \{x \in \mathbb{R}^n, |x| = 1\}$ . For balls the representations (2) or (3) are highly non-unique, since any  $y \in \partial_e\mathcal{M}$  can show up on the right hand side with positive weight.

The essential difference between the state space of classical mechanics and that of quantum mechanics is:

- The state space of classical mechanics is an (infinite dimensional) simplex with a unique representation (2).
- The state space of quantum mechanics is not a simplex, its structure is closer to that of a ball. The decompositions (2) or (3) into extremal elements, i.e. pure states as explained below, are highly non-unique.

The state space of classical mechanics is the set of all probability measures  $d\mu(\xi)$  on the phase space  $\Xi \subset \mathbb{R}^n$  of a given system. The extremal boundary of that set is given by all Dirac measures  $\{\delta_\eta(\xi) d^n \xi = \delta(\xi - \eta) d^n \xi | \eta \in \Xi\}$ . The representation (2) is the identity  $d\mu(\xi) = \int \delta_\eta(\xi) d\mu(\eta)$ , which leads again to the originally given measure. For the mathematical proof that the space of probability measures is an infinite dimensional simplex with a unique Choquet representation, see [11,12].

We have already seen that the state space of quantum mechanics is convex. The boundary of this set is formed by those statistical operators which have at least one eigenvalue zero. The condition for extremal elements, i.e.  $W = \frac{1}{2}W_1 = \frac{1}{2}W_2$  has to imply  $W_1 = W_2$ , is met by all rank one projection operators  $W \in \mathcal{P}(\mathcal{H})$ . The decomposition (1) is therefore

the Choquet representation (3) of an element of a convex set. As a consequence of the superposition principle this representation is highly non-unique, see e.g. § 2.3 of [13]. As explicit example we consider the state space of a spin- $\frac{1}{2}$  system, i.e.  $\mathcal{D}(\mathbb{C}^2)$ . Since all statistical operators on  $\mathbb{C}^2$  with one eigenvalue zero are rank one projectors, the boundary of the state space coincides with the extreme boundary. More explicitly, any operator  $\rho \in \mathcal{D}(\mathbb{C}^2)$  can be represented with a real polarization vector  $\vec{p}$  in the unit ball  $B^3 = \{\vec{p} \in \mathbb{R}^3 \mid |\vec{p}| \leq 1\}$  as

$$\widehat{\rho}(\vec{p}) := \frac{1}{2} (\mathbf{1} + \vec{\sigma}\vec{p}) \tag{4}$$

where  $\mathbf{1}$  is the unit 2x2 matrix and  $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  are the Pauli matrices. Moreover, a simple calculation leads to the identity  $\|\rho(\vec{p}_1) - \rho(\vec{p}_2)\|_1 = |\vec{p}_1 - \vec{p}_2|$ . Hence the topology of  $\mathcal{D}(\mathbb{C}^2)$  given by the trace norm agrees with the euclidean metric of the polarization vectors. The set  $\mathcal{D}(\mathbb{C}^2)$  is therefore isomorphic to the unit ball  $B^3$ . The extremal elements corresponding to  $\vec{p} \in S^2$ , are exactly the projection operators of the representation (1). Since  $\lambda_1\rho(\vec{p}_1) + \lambda_2\rho(\vec{p}_2) = \rho(\lambda_1\vec{p}_1 + \lambda_2\vec{p}_2)$  if  $\lambda_{1,2} \geq 0, \lambda_1 + \lambda_2 = 1$ , the representation (1) corresponds therefore to the Choquet representation (3) of the unit ball  $B^3$ . The non-uniqueness of that representation is therefore obvious.

In higher dimensions the geometrical picture of  $\mathcal{D}(\mathcal{H})$ ,  $\dim\mathcal{H} \geq 3$ , is more complicated. The boundary is much larger than the extremal boundary, but the arbitrariness of the representation (1) remains. To see this one can choose any two dimensional subspace of the spectral decomposition of  $W$  with at least one non-zero eigenvalue. Then this part of the spectral representation corresponds up to normalization to  $\mathcal{D}(\mathbb{C}^2)$ , and we can modify it with all the arbitrariness seen above.

### 3. Superselection rules

The arbitrariness of the decomposition (1) originates in the superposition principle. In quantum mechanics, especially in quantum field theory, the superposition principle can be restricted by superselection rules. Here we cannot discuss the arguments to establish such rules, for that purpose see e.g. [5,6], and also chap.6 of [10], or to refute them, see e.g. [14]. Here we only investigate the consequences for the structure of the state space. In a theory with superselection rules like the charge superselection rule, the Hilbert space  $\mathcal{H}$  splits into orthogonal superselection sectors  $\mathcal{H}_m, m \in \mathbf{M}$ , such that  $\mathcal{H} = \oplus_m \mathcal{H}_m$ . Pure states with charge  $m$  (in appropriate normalization) are then represented by vectors in  $\mathcal{H}_m$ , and superpositions of vectors with different charges have no physical interpretation. The projection operators  $P_m$  onto the orthogonal subspaces  $\mathcal{H}_m$  satisfy  $P_m P_n = \delta_{mn}$  and  $\sum_m P_m = I$ . The set of states is reduced to those statistical operators which satisfy  $P_m W = W P_m$  for all projection operators  $P_m, m \in \mathbf{M}$ . The state space of the system is then  $\mathcal{D}^S = \{W \in \mathcal{D}(\mathcal{H}) \mid W P_m = P_m W, m \in \mathbf{M}\}$ , and all statistical operators satisfy the identity

$$W = \sum_m P_m W P_m. \tag{5}$$

A mathematical equivalent statement is that all observables of such a theory commute with the projection operators  $P_m, m \in \mathbf{M}$ . Superselection rules of this type will be called

“kinematical superselection rules” to contrast them against the “dynamically induced superselection rules” to be discussed in the following section.

As a consequence of (5) the extremal boundary of  $\mathcal{D}^S$  decomposes into  $\mathcal{P}^S = \cup_m \mathcal{P}_m^S$  with  $\mathcal{P}_m^S = \{P^1 \in \mathcal{P}(\mathcal{H}) | P^1 \mathcal{H} \subset \mathcal{H}_m\}$ . The decomposition of a statistical operator  $W \in \mathcal{D}^S$  into pure states now reads (here the sum over  $i$  can be substituted by an integral over the set  $\mathcal{P}_m^S$ )  $W = \sum_{m,i} \lambda_{m,i} P_{m,i}^1$  with  $P_{m,i}^1 \in \mathcal{P}_m^S$  and probabilities  $\lambda_{m,i} \geq 0$ ,  $\sum_{m,i} \lambda_{m,i} = 1$ . Since the representation of an element of  $\mathcal{D}(\mathcal{H}_m)$  by pure states is not uniquely given (if  $\dim \mathcal{H}_m \geq 2$ ), the right hand side of this decomposition is again highly non-unique. But as a consequence of the structure of the state space  $\mathcal{D}^S$  the total probability with respect to any of the subsets  $\mathcal{P}_m^S$ , i.e.  $\sum_i \lambda_{m,i}$ , has no ambiguity since  $\sum_i \lambda_{m,i} = \text{tr} W P_m$ , where the right hand side is uniquely defined. Hence superselection rules allow to speak about objective “properties” of a quantum system. These properties show up with the probabilities  $\text{tr} W P_m$  irrespective of what other specifications (measurements) are made. In the language of observables these properties are represented by the selfadjoint operators  $P_m$ , which commute with all observables of the system. The importance of superselection rules for the transition from quantum probability to classical probability is obvious. But there remains an essential problem: Only very few superselection rules can be found in quantum mechanics that are compatible with the mathematical structure and with experiment. A satisfactory solution to this problem is the emergence of effective superselection rules induced by the interaction with the environment.

#### 4. Dynamics of subsystems

In the following we consider an “open system”, i.e. a system  $S$  which interacts with an “environment”  $E$ , such that the total system  $S + E$  satisfies the usual Hamiltonian dynamics. The system  $S$  is singled out by the fact that all observations refer only to this subsystem. The Hilbert space  $\mathcal{H}_{S+E}$  of the total system  $S + E$  is the tensor space  $\mathcal{H}_S \otimes \mathcal{H}_E$  of the Hilbert spaces for  $S$  and for  $E$ . We assume that the only observables at our disposal are the operators  $A \otimes I_E$  where  $A \in \mathcal{B}(\mathcal{H}_S)$  is an arbitrary bounded selfadjoint operator on  $\mathcal{H}_S$ . If the state of the total system is  $W \in \mathcal{D}(\mathcal{H}_{S+E})$ , then all expectation values  $\text{tr} W (A \otimes I_E)$  can be calculated from the reduced statistical operator  $\rho = \text{tr}_E W$  which is an element of  $\mathcal{D}(\mathcal{H}_S)$  defined such that the expectation values satisfy  $\text{tr}_S \rho A = \text{tr}_{S+E} W (A \otimes I_E)$ . Since all information about a physical subsystem is given by a statistical operator, we shall here refer to the statistical operator as the “state” of the subsystem. (The state of the total system cannot be recovered from these “states” of its subsystems.)

As mentioned above we assume the usual Hamiltonian dynamics for the total system, i.e.  $W(t) = U(t)W(0)U^+(t)$  with the unitary group  $U(t)$ , generated by the total Hamiltonian. Except for the trivial case that  $S$  and  $E$  do not interact, the dynamics of the reduced statistical operator  $\rho(t) = \text{tr}_E U(t)W(0)U^+(t)$  is no longer unitary. It is the purpose of the second part of this talk to evaluate this dynamics in some detail. The essential result is that this dynamics can produce effective superselection sectors, i.e.

$$\rho(t) \cong \sum_m P_m \rho(t) P_m \tag{6}$$

in sufficiently short time with a set of projection operators  $P_m, m \in \mathbf{M}$ , which correspond to a superselection structure (5). In §5 we shall give more precise formulations of (6). The

suppression of the off-diagonal terms of the statistical operator in (6) is essential for the emergence of classical properties in quantum mechanics. This suppression can never be understood by semiclassical approximations alone.

5. Solvable models

The statement (6) is so far rather vague since it does not specify the asymptotics. The following examples show what type of asymptotics is possible in principle. For a restricted class of models an estimate

$$\|\rho(t) - \sum_n P_n \rho(t) P_n\|_2 = \left\| \sum_{m \neq n} P_m \rho(t) P_n \right\|_2 \leq C_\gamma (1 + \delta |t|)^{-\gamma} \quad (7)$$

with the Hilbert–Schmidt norm is possible for arbitrary  $\rho(0) \equiv \rho \in \mathcal{D}(\mathcal{H}_S)$ . The constants  $\gamma > 0$ ,  $\delta > 0$  and  $C_\gamma > 0$  do not depend on  $\rho$ . Since one can achieve large values of  $\gamma$  and/or small values of the constant  $C_\gamma$ , these dynamically induced superselection sectors  $P_n \mathcal{H}$  cannot be distinguished practically from the kinematical superselection sectors (5). For a subclass of these models one can even derive an estimate of the type (7) with the stronger trace norm. A consequence of (7) is that the transition between a superselection sector  $P_n \mathcal{H}$  and its complement  $\widehat{P}_n \mathcal{H} := (I - P_n) \mathcal{H}$  is suppressed by  $\|\widehat{P}_n \rho(t) P_n\|_2 \leq C_\gamma (1 + |t|)^{-\gamma}$  again with the Hilbert–Schmidt norm. If  $P$  is a projection operator with finite rank  $N$  on a subspace of  $P_n \mathcal{H}$ , i.e.  $P_n P = P P_n = P$ , then  $\widehat{P}_n \rho(t) P$  is a traceclass operator with  $\|\widehat{P}_n \rho(t) P\|_1 \leq \sqrt{N} C_\gamma (1 + |t|)^{-\gamma}$ , and the transition between  $P \mathcal{H}$  and all other sectors  $P_m \mathcal{H}$ ,  $m \neq n$ , is uniformly suppressed in trace norm. The investigation of the models also shows that unfortunately such simple estimates are rather unstable against slight modifications of the models.

The models have the following structure. The Hilbert space is  $\mathcal{H}_{S+E} = \mathcal{H}_S \otimes \mathcal{H}_E$ . The total Hamiltonian has the form

$$H = H_S \otimes I_E + I_S \otimes H_E + V_S \otimes V_E + V \quad (8)$$

where  $H_S$  is the Hamiltonian of S,  $H_E$  is the Hamiltonian of E,  $V_S \otimes V_E$  is the interaction term between S and E with selfadjoint operators  $V_S$  on  $\mathcal{H}_S$  and  $V_E$  on  $\mathcal{H}_E$ , and  $V$  is a possible additional scattering potential. For all models we assume that

1. The operators  $H_E$  and  $V_E$  commute,  $[H_E, V_E] = 0$ , hence  $[I_S \otimes H_E, V_S \otimes V_E] = 0$ .
2. The operator  $V_S$  has a pure point spectrum

$$V_S = \sum_m \lambda_m P_m \quad (9)$$

where the projection operators  $P_m$  form a complete set,  $P_m P_n = \delta_{m,n} P_n$ , and  $\sum_m P_m = I$ . We assume that the eigenvalues  $\lambda_m$  are separated with the lower bound  $|\lambda_m - \lambda_n| \geq \delta > 0$  if  $m \neq n$ .

3. The operator  $V_E$  has an (absolutely) continuous spectrum.

Remark. The continuous spectrum of  $V_E$  is needed to obtain simple estimates for  $t \rightarrow \infty$ . But one could also allow an operator with point spectrum (as done in [8]), if the spacing of the eigenvalues is sufficiently small. Then the norm in (7) is an almost periodic function and the inequality is only correct for a finite time interval  $0 \leq t \leq T$ . But  $T$  can be large enough for all practical purposes.

### 5.1 The Araki–Zurek model

The first solvable models to discuss the reduced dynamics have been given by Araki [7] and Zurek [8], and the following construction is essentially based on these papers. In addition to the specifications made above, we demand:

4. The operators  $H_S$  and  $V_S$  commute, the potential  $V$  vanishes, i.e.  $[H_S, V_S] = 0 = V$ .

For an originally factorizing state  $W = \rho \otimes \omega$  we calculate with  $U(t) = \exp(-iHt)$

$$\rho(t) = \text{tr}_E U(t) W U^+(t) = e^{-iH_S t} \sum_{m,n} P_m \rho P_n e^{iH_S t} \chi_{m,n}(t) \quad (10)$$

with  $\chi_{m,n}(t) = \text{tr} (e^{-i(\lambda_m - \lambda_n)V_E t} \omega)$ , see e.g. § 7.6 of [10]. The trace  $\chi_{m,n}(t)$  vanishes for  $|t| \rightarrow \infty$  if  $m \neq n$  since  $V_E$  has an absolutely continuous spectrum. Under additional smoothness assumptions on  $\omega$  (which do not restrict  $\rho \in \mathcal{D}(\mathcal{H}_S)$ ) we derive for  $\chi(t) := \text{tr} (e^{-iV_E t} \omega)$  the estimate  $|\chi(t)| \leq C_\gamma (1 + |t|)^{-\gamma}$ . Here  $\gamma$  can be arbitrarily large if  $\omega$  is a sufficiently differentiable function in the spectral representation of  $V_E$  (and vanishes at the boundary points of the spectrum). This estimate leads to the upper bound

$$|\chi_{m,n}(t)| \leq C_\gamma (1 + \delta |t|)^{-\gamma} \quad (11)$$

if  $|\lambda_m - \lambda_n| \geq \delta > 0$ , and we obtain the estimate (7). If we have only a finite number of eigenvalues in (9), a bound of the type (7) with the stronger trace norm can be derived.

This result depends on the reference state  $\omega$  only via the decrease of  $\chi(t)$ . We could have chosen a more general initial state  $W = \sum_\mu \rho_\mu \otimes \omega_\mu \in \mathcal{D}(\mathcal{H}_{S+E})$  with  $\rho_\mu \in \mathcal{T}(\mathcal{H}_S)$ ,  $\omega_\mu \in \mathcal{D}(\mathcal{H}_E)$ . The operators  $\rho_\mu$  need not be positive separately, but we have of course  $\rho = \sum_\mu \rho_\mu \in \mathcal{D}(\mathcal{H}_S)$ . Then (7) is still valid if  $\sum_\mu |\text{tr} (e^{-i(\lambda_m - \lambda_n)V_E t} \omega_\mu)|$  satisfies a uniform estimate (11). Hence the emergence of the superselection sectors  $P_n \mathcal{H}_S$  is not sensitive to the initial conditions.

### 5.2 A spin model

So far we have made the rather restrictive assumption  $V = [H_S, V_S] = 0$  such that the interaction commutes with the free Hamiltonian  $H_S \otimes I_E$ . The following model has still  $V = 0$ , but  $[H_S, V_S]$  does no longer vanish.

The Hilbert space of the model is  $\mathcal{H}_{S+E} = \mathcal{H}_S \otimes \mathcal{H}_E$  with  $\mathcal{H}_S = \mathbb{C}^2$  and  $\mathcal{H}_E = \mathcal{L}^2(\mathbb{R})$ . The Hamiltonian has the form (8) with  $V = 0$  and the following specifications

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$$H_S \psi = (\vec{a} \vec{\sigma}) \psi, \text{ with } \vec{a} \in \mathbf{R}^3, \vec{\sigma} \text{ Pauli matrices, } \psi \in \mathbf{C}^2,$$

$$H_E f(x) = bx^2 f(x) \text{ with a positive constant } b > 0, f(x) \in \mathcal{L}^2(\mathbf{R}),$$

$$V_S = \lambda \sigma_3, \text{ with a real coupling parameter } \lambda,$$

$$V_E f(x) = xf(x), f(x) \in \mathcal{L}^2(\mathbf{R}).$$

The statistical operator of the spin- $\frac{1}{2}$  system is a spin density matrix (4) with a polarization vector  $\vec{p} \in B^3$ . For the total system we assume an initial state  $W = \hat{\rho}(\vec{p}) \otimes \omega$  where  $\omega$  is a statistical operator on  $\mathcal{L}^2(\mathbf{R})$  with a smooth integral kernel  $\omega(x, y)$ . The time evolution  $U(t) = \exp(-iHt)$  with the total Hamiltonian (8) then leads to the state  $U(t)WU^+(t)$ . The diagonal part of the integral kernel of this statistical operator is  $(U(t)WU^+(t))(x, x) = \omega(x, x) \hat{\rho}(R_x(t)\vec{p})$ . Here  $R_x(t)$  is the rotation induced by the  $SU(2)$  matrix part of  $U(t)$ , i.e.  $\exp(-iht)$ , with  $h = (\vec{a}\vec{\sigma}) + \lambda x \sigma_3$ . The reduced statistical operator  $\rho(t)$  of the spin- $\frac{1}{2}$  system is calculated as  $\rho(t) = \text{tr}_E U(t)WU^+(t) = \int dx \omega(x, x) \hat{\rho}(R_x(t)\vec{p})$ . For the initial state  $\rho = \text{tr}_E W = \hat{\rho}(\vec{p})$  the reduced dynamics then leads to a statistical operator  $\rho(t)$ , which asymptotically approaches  $\hat{\rho}(\vec{q})$ , where the polarization vector  $\vec{q}$  is given by the linear mapping

$$\vec{q} = M\vec{p} := \int dx \omega(x, x) \vec{n}(x) (\vec{p} \vec{n}(x)) \quad (12)$$

with the axis  $\vec{n}(x)$  of the rotation  $R_x(t)$ . Under appropriate conditions for the initial state  $\omega$  of the environment, the difference  $\rho(t) - \hat{\rho}(\vec{q})$  can be uniformly estimated by  $\|\rho(t) - \hat{\rho}(\vec{q})\|_1 \leq c(1 + |t|)^{-\gamma}$ . The mapping (12) is a symmetric contraction on  $\mathbf{R}^3$ . We can distinguish two cases:

1. If  $\vec{a} \parallel \vec{e}_3$  the mapping (12) reduces to  $M\vec{p} = \vec{e}_3 (\vec{e}_3 \vec{p})$ , and we obtain the results discussed in §5.1. The condition  $|M\vec{p}| = |\vec{p}|$  is satisfied for  $\vec{p} \parallel \vec{e}_3$ , and only in this case  $\rho(t)$  is not affected by the decoherence.
2. If  $\vec{a}$  has components orthogonal to  $\vec{e}_3$ , also the direction of  $M\vec{p}$  depends on  $\vec{p}$ , and  $|M\vec{p}| < |\vec{p}|$  holds for all vectors  $\vec{p} \neq \vec{0}$ .

In the second case there are no projection operators which commute with all operators  $\hat{\rho}(M\vec{p}), \vec{p} \in B^3$ . A superselection rule of the type (6) can hold in some approximative sense only if  $a_1^2 + a_2^2 \ll a_3^2$ . In that case  $M\vec{p}$  has very small components orthogonal to  $\vec{e}_3$ , and the off-diagonal matrix elements of the operators  $\hat{\rho}(M\vec{p}), \vec{p} \in B^3$ , are negligible.

### 5.3 Models with scattering

We assume again a Hamiltonian (8) with  $[H_S, V_S] = 0$  as in the Araki-Zurek model, but now with an additional scattering potential  $V$  defined on the full Hilbert space  $\mathcal{H}_{S+E} = \mathcal{H}_S \otimes \mathcal{H}_E$ . Under appropriate conditions on  $V$  the wave operator exists as strong limit  $\Omega = \lim_{t \rightarrow \infty} e^{iHt} e^{-iH_0 t}$  on  $\mathcal{H}_{S+E}$ , where  $H_0$  is the Hamiltonian of the Araki-Zurek model,  $H_0 = H_S \otimes I_E + I_S \otimes H_E + V_S \otimes V_E$ . (We assume for simplicity that there are no bound states such that  $\Omega^+ = \Omega^{-1}$ .) Then the time evolution  $U(t) = \exp(-iHt)$

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behaves asymptotically as  $U_0(t)\Omega^+$  with  $U_0(t) = \exp(-iH_0t)$ . More precisely we have for all  $W \in \mathcal{D}(\mathcal{H}_{S+E})$

$$\lim_{t \rightarrow \infty} \|U(t)WU^+(t) - U_0(t)\Omega^+W\Omega U_0^+(t)\|_1 = 0 \quad (13)$$

in the trace norm. For initial states  $W = \rho(0) \otimes \omega$  with smooth  $\omega$ , and for sufficiently regular scattering potentials the statistical operator  $\Omega^+W\Omega$  has only smooth contributions in  $\mathcal{T}(\mathcal{H})$ , and the reduced trace  $\text{tr}_E U_0(t)\Omega^+W\Omega U_0^+(t)$  yields the induced superselection sectors  $P_m \mathcal{H}_S$  of §5.1. But then (13) implies for  $\rho(t) = \text{tr}_E U(t)WU^+(t)$  the asymptotics  $\lim_{t \rightarrow \infty} \|\rho(t) - \sum_m P_m \rho(t) P_m\|_2 = 0$ . Hence  $\rho(t)$  has again the induced superselection sectors which originate from the spectrum (9) of  $V_S$ . But in contrast to (7) this bound is not uniform in the initial state  $\rho(0)$ , since scattering does not allow a uniform estimate for (13).

#### 5.4 Concluding remarks

The investigation of the models proves that the uniform emergence (7) of superselection sectors is consistent with the mathematical rules of quantum mechanics. But this result depends on rather restrictive assumptions on the Hamiltonian. For more realistic models we can nevertheless expect a strong quantitative suppression of the off-diagonal elements of the statistical operator, as already proposed in [8].

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