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Nonlinear effects on Nielsen–Olesen instability

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Abstract. We have obtained an analytical solution of the nonlinear Nielsen–Olesen mode, which is known to be linearly unstable. We believe that our solution may be a nonlinear saturated state of the instability. It is different from the flux tube structure of chromomagnetic field as proposed by the Copenhagen group [1–3]. We have reanalysed their calculations, including a term, which they have neglected, and find it consistent with our result.

Keywords. $SU(2)$ Yang–Mills fields; Nielsen–Olesen's instability; nonlinear state; flux tube structure of QCD vacuum.

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Non-perturbative confining nature of quantumchromodynamics (QCD) vacuum is believed to be due to the non-Abelian nature of the theory. One of the results in this direction was obtained by Savvidy [4] who showed that uniform chromomagnetic field in one color and one space direction has lower energy than perturbative vacuum and that QCD vacuum may be a ferromagnetic state [5]. However, this state breaks Lorentz and rotational invariance and may not be a true vacuum state. Later, Nielsen and Olesen showed that Savvidy state is unstable to color perturbations in other directions [6]. It was also shown that, Savvidy state is not the actual QCD vacuum and a lower energy state may be obtained by stabilizing the instability [2, 3]. They used the unstable solution to evaluate the energy density and minimized it to determine the unknown parameters of the solution. They chose a special delta function form for the Fourier amplitudes to obtain flux tube structure of chromomagnetic field. They argued, based on quantum mechanics, that further lowering of energy is possible by excitation of vibration and rotational degrees of freedom of flux tubes and hence obtained the 'spaghetti of flux tube' structure of QCD vacuum [1].

We note that the 'spaghetti of flux tube' structure of QCD vacuum is based on perturbative analysis. The complete nonlinearity of QCD is not exploited. It is this nonlinearity which we want to investigate here. Our approach is based on the well-known fact that as the perturbation grows due to instability, nonlinear effects become important and the instability saturates [7]. One such state is obtained by solving the corresponding static nonlinear equation. In our problem it may be the flux tube structure of the Copenhagen group. However, we find that the static nonlinear solution is not the flux tube structure but a zero field state. Thus Nielsen–Olesen instability (NOI) tries to get rid of the uniform chromomagnetic field (H_0) completely and thereby lowers the energy by

$H_0^2/2$. However, nontrivial vector potentials, which correspond to nonlinear version of Copenhagen Yang–Mills field condensates, do exist. These solutions have periodic structures in one space direction.

Yang–Mills equations of motion in $SU(2)$ is

$$\partial_\mu G_a^{\mu\nu} + g\epsilon_{abc}A_{\mu b}G_c^{\mu\nu} = 0, \tag{1}$$

where the field tensor

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\epsilon_{abc}A_b^\mu A_c^\nu,$$

a, b, c are color indices which take values 1, 2, 3 and Lorentz indices $\mu, \nu = 0, 1, 2, 3$ with metric $(1, -1, -1, -1)$. ϵ_{abc} is antisymmetric Levi–Civita tensor. In terms of a new set of variables [6, 8] $A^\mu \equiv gA_3^\mu$ and $X^\mu \equiv g\frac{1}{\sqrt{2}}(A_1^\mu + iA_2^\mu)$, eq. (1) reduces to

$$\partial_\mu F^{\mu\nu} - i[\partial_\mu(X^{\mu*}X^\nu) + X_\mu(D^\nu X^\mu)^* - X_\mu(D^\mu X^\nu)^* - [c.c]] = 0, \tag{2}$$

$$D_\mu D^\mu X^\nu - 2iX_\mu F^{\mu\nu} + [(X.X)X^{\nu*} - (X.X)^* X^\nu] = 0, \tag{3}$$

where $[c \cdot c]$ means complex conjugate,

$$D^\mu \equiv \partial^\mu + iA^\mu, \; F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu.$$

and we used the gauge $D_\mu X^\mu = 0$.

Savvidy state of uniform chromomagnetic field solution, namely, $A^\mu = H_0 x\hat{z}$ and $X^\mu = 0$ is a solution of the above equations. Linearizing the above eqs (2–3) about the Savvidy state leads to Nielsen–Olesen instability. Using this perturbative solution Ambjorn and Olesen constructed the flux tube structure by minimizing the energy density $\varepsilon = (G_3^{12})^2/2$ with respect to few parameters [2, 3], where

$$gG_3^{12} = F^{12} + |W|^2, \tag{4}$$

and all other components of $G_a^{\mu\nu}$ are zero. W is the color field X^μ corresponding to the unstable mode. They took F^{12} as unperturbed solution H_0 and since W is a periodic function of x and y, one can get flux tube structure of the chromomagnetic field. Periodicity in W is coming because of the choice of Fourier amplitude as delta function and the periodicity and a constant amplitude are determined by minimizing ε.

Our approach is different. We solve the full nonlinear equations (2–3) in the static limit. Since we are interested in the unstable mode we look for the solution such that $X_x = -iX_y = W/\sqrt{2}$. Using the gauge $D_\mu X^\mu = 0$, one can integrate the eq. (2) [9] to get

$$F^{\mu\nu} = -iX^\mu X^{\nu*} + [c \cdot c] + C, \tag{5}$$

where C is a constant. Or,

$$F^{12} = -|W|^2 + C, \tag{6}$$

W is a complex function and may be written as $W = |W|e^{i\chi}$. The gauge condition $D_\mu X^\mu = 0$ reduces to

$$[\partial_x + i(A_x + \partial_x\chi)]|W| + [\partial_y + i(A_y + \partial_y\chi)]|W|i = 0. \tag{7}$$

Redefining

$$\tilde{A}_x \equiv A_x + \partial_x\chi; \quad \tilde{A}_y \equiv A_y + \partial_y\chi, \tag{8}$$

and separating into real and imaginary parts of the eq. (7) we get

$$\tilde{A}_x = \partial_y \ln|W|; \quad \tilde{A}_y = -\partial_x \ln|W| \tag{9}$$

Note that eq. (8) is nothing but the gauge transformation of A^μ with gauge parameter equal to the argument of W field, χ, which leaves $F^{\mu\nu}$, eq. (6), invariant. Using the definition, $F^{12} = -\partial_x A_y + \partial_y A_x = -\partial_x \tilde{A}_y + \partial_y \tilde{A}_x$, eq. (6) gives

$$(\partial_x^2 + \partial_y^2)\ln|W| = -|W|^2 + C, \tag{10}$$

or

$$(\partial_x^2 + \partial_y^2 - \tilde{A}_x^2 - \tilde{A}_y^2)|W| = -|W|^3 + C|W|. \tag{11}$$

From eq. (3) we also have

$$(-\partial_x^2 - \partial_y^2 + \tilde{A}_x^2 + \tilde{A}_y^2 + 2F^{12})|W| = -|W|^3. \tag{12}$$

Comparing the above two equations and using eq. (6) one can see that either $|W| = 0$ or $C = 0$. For $|W| = 0$, vacuum can have nonzero C which is like Savvidy's uniform chromomagnetic field. But for finite $|W|$, C must be zero. Therefore, linear calculation of Ambjorn and Olesen with both C and $|W|$ finite, is not allowed. Hence the instability may be driving the system to a state with $C = 0$ and $|W| \neq 0$. From eqs (4) and (6) it follows that $G_a^{\mu\nu} = 0$. Hence the instability brought down the finite $G_a^{\mu\nu}$ state (finite energy) to a state with zero field (zero energy) state.

In order to find the solution W, we need to solve the nonlinear equation (10) with $C = 0$. Substituting $|W| = W_0 e^\phi$, it may be written as

$$(\partial_x^2 + \partial_y^2)\phi = -W_0^2 e^{2\phi}, \tag{13}$$

where W_0 is a constant. This is a well-known nonlinear equation in the study of vertices in fluid and plasma physics [10] and has an analytic solution

$$\phi = -\ln(\cosh(\alpha x) + \epsilon\cos(\alpha y)). \tag{14}$$

or

$$|W| = \frac{W_0}{\cosh(\alpha x) + \epsilon\cos(\alpha y)}, \tag{15}$$

where $\epsilon = \sqrt{1 - W_0^2/\alpha^2}$. It is periodic in y with wave number α. Other fields, from eq. (9), are

$$\tilde{A}_x = \frac{\sqrt{\alpha^2 - W_0^2}\sin(\alpha y)}{\cosh(\alpha x) + \epsilon\cos(\alpha y)} \tag{16}$$

$$\tilde{A}_y = \frac{\alpha\sinh(\alpha x)}{\cosh(\alpha x) + \epsilon\cos(\alpha y)} \tag{17}$$

For $\alpha^2 = W_0^2, |W| = W_0\,\mathrm{sech}(W_0 x)$ which we obtained in our earlier one-dimensional calculation [11]. Comparing this solution near $x = 0$ with linear unstable mode, which is a Gaussian, we may take $W_0^2 = H_0$ [11]. Solution exists only for $\alpha^2 \geq W_0^2 = H_0$. Therefore, this nonlinear state is such that the scale length of inhomogeneity of field in

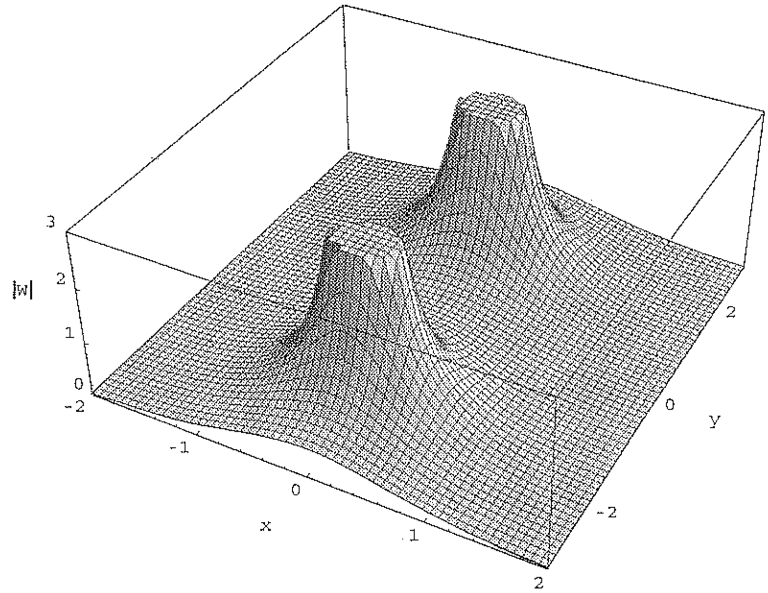

Figure 1. Surface plot of $|W|$ from eq. (14) for $H_0 = 1$ and $\alpha^2 = 2H_0$.

space is less than the inverse of square root of the field strength of the condensate and hence stable against Nielsen–Olesen instability. The idea of having $\alpha^2 \geq H_0$ to stabilize NOI was also used by the Copenhagen group where they had constructed solution having similar property.

The plots of above fields are shown in figures 1 to 3 for $W_0^2 = H_0 = 1$ and $\alpha^2 = 2H_0$ as an example. We may take $\sqrt{H_0} = \Lambda_s \approx 240$ MeV [12], Savvidy's QCD scale, which is close to 1 fm^{-1}, but α^2 is arbitrary and greater than H_0. They have periodic structure in one direction (say y) only, in contrast to the linear solution of Ambjorn and Olesen for W, which is periodic both in x and y. However, we have noticed in our 1D calculation [11] that if one includes other modes as well, in the analysis one can get periodicity in x direction also. At present it is not clear whether we can get analytical nonlinear periodic solution both in x and y. The solution of eq. 10 in polar coordinates (r, θ) is given by

$$|W| = \frac{W_0 r_0}{r[\cosh(\alpha \ln(r/r_0)) + \epsilon \cos(\alpha\theta)]}, \qquad (18)$$

where α is a constant and $\epsilon = \sqrt{1 - W_0^2 r_0^2/\alpha^2}$. For single valuedness, α should be an integer excluding zero. The solution does not exist for α equal to zero. W_0 and r_0 are constants which determine strength and size of the flux tube of magnetic field $F^{\mu\nu}$. The surface plot of this solution is shown in figures 4 and 5 for α equal to 4 and 6 as an example. These plots show some similarity to the condensate solutions of the Copenhagen group for QCD [3] and electroweak theory [13]. Our nonlinear periodic solution for W is such that chromomagnetic field ($G_a^{\mu\nu}$) is zero. We get flux tube structure for $F^{\mu\nu}$ but not for $G_a^{\mu\nu}$. It looks contrary to the results of the Copenhagen group [3]. However, in their calculation perturbative contribution to F^{12} was not taken. That is,

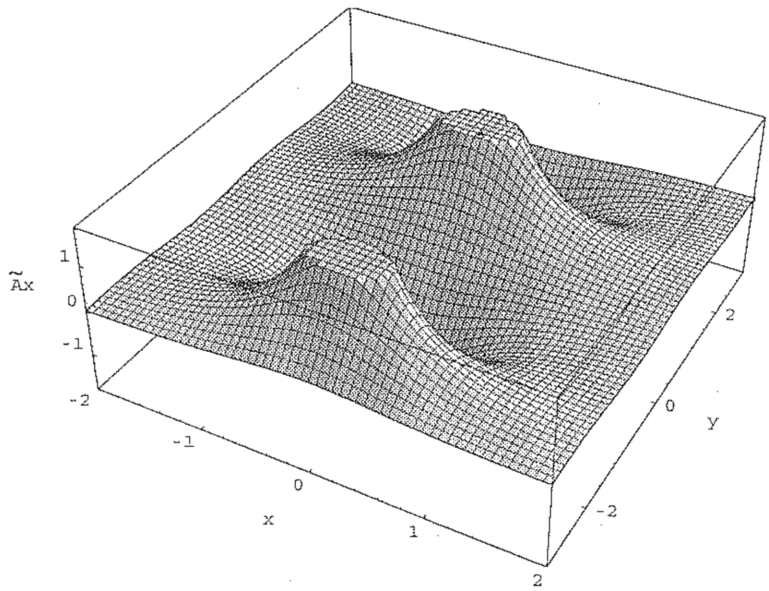

Figure 2. Surface plot of $\tilde{A}_x$ from eq. (15) for $H_0 = 1$ and $\alpha^2 = 2H_0$.

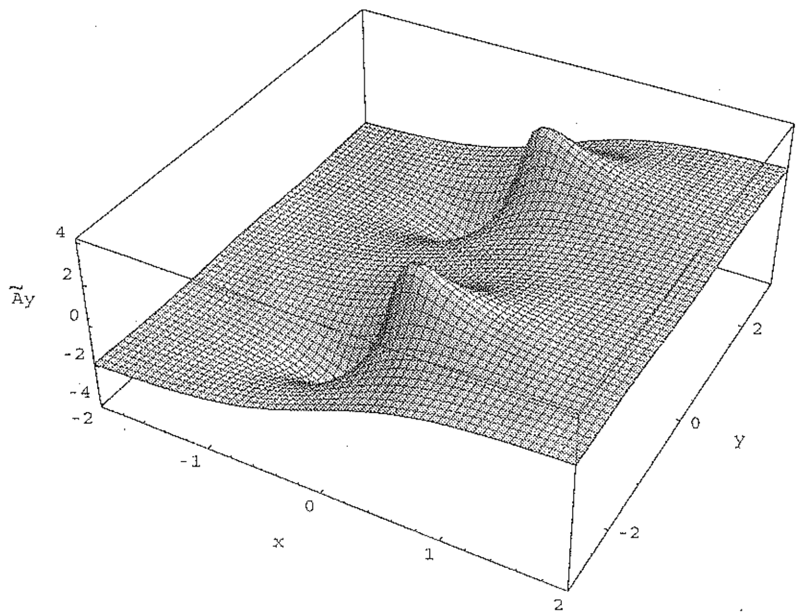

Figure 3. Surface plot of $\tilde{A}_y$ from eq. (16) for $H_0 = 1$ and $\alpha^2 = 2H_0$.

eq. (4), on perturbation around the Savvidy state ($F^{12} = -H_0$ and $W = 0$) gives

$$gG_3^{12} = F_{(0)}^{12} + aF_{(1)}^{12} + a^2F_{(2)}^{12} + a^2|W_{(1)}|^2 + \cdots, \tag{19}$$

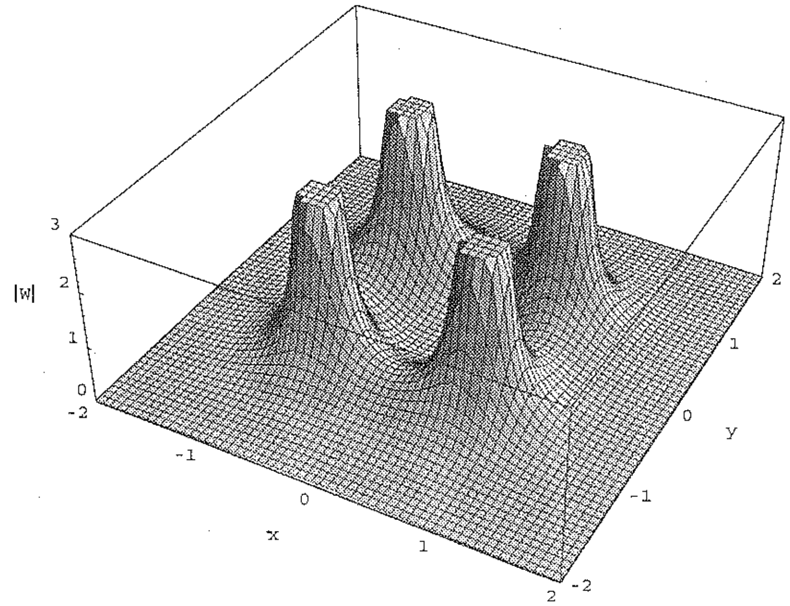

Figure 4. Surface plot of $|W|$ in polar coordinates from eq. (17) for $H_0 = r_0 = 1$ and $\alpha = 4$.

where a is the perturbation parameter and lower subscript with bracket indicates the order of perturbation. It is easy to see that zeroth order and first order equations for $F^{\mu\nu}$, from eq. (2), are same and hence we may take $F^{12}_{(1)} = 0$. But $F^{12}_{(2)}$ is nonzero and is driven by $W_{(1)}$ field. The second order equation for $F^{\mu\nu}$, from eq. (2), may be integrated [11] to get $F^{12}_{(2)} = -|W_{(1)}|^2 + D$ in one-dimensional problem of Copenhagen group, where D is the integration constant. Substituting for $F^{12}_{(2)}$ in eq. (19), we get $gG^{12}_3 = -H_0 + Da^2 + \cdots$. The space dependent in G^{12}_3 field, coming from $|W_{(1)}|$, just cancels because of $F^{12}_{(2)}$ correction, which was not taken in account by Copenhagen group and G^{12}_3 is constant. Therefore, there will not be any flux tube structure of chromomagnetic fields even though there is a nonzero periodic condensate solution $W(x, y)$. Further more, from the solution $|W_{(1)}| = \sqrt{H_0}\, e^{(-H_0 x^2/2)}$ we have $|W_{(1)}|^2 = H_0$ at $x = 0$. Hence integration constant D may be taken as H_0 to make $F^{12}_{(2)} = 0$ at $x = 0$. This ensures that $F^{12} = -H_0$, unperturbed value, at $x = 0$ as demanded by the exact solution eq. (17) for one-dimensional problem. Thus we get $gG^{12}_3 = -H_0(1 - a^2 + \cdots)$ and is reduced from it's original value $-H_0$.

In summary, we have found analytical solutions of the static nonlinear $SU(2)$ Nielsen–Olesen mode. These solutions may be a nonlinearly saturated state of Nielsen–Olesen instability. Magnetic field $F^{\mu\nu}$ has flux tube structure periodic in one of the directions perpendicular to it and hence has some similarities with perturbative as well as numerical calculations of the Copenhagen group. However, the field tensor $G^{\mu\nu}_a$ is zero for these solutions and hence the picture of QCD vacuum as 'the spaghetti of flux tubes' of chromomagnetic field [1] may not be correct. We arrived at the same conclusion by repeating the perturbative calculations of the Copenhagen group in a consistent way, i.e., by including a very important term in evaluating $G^{\mu\nu}_a$ which they had neglected. Thus the

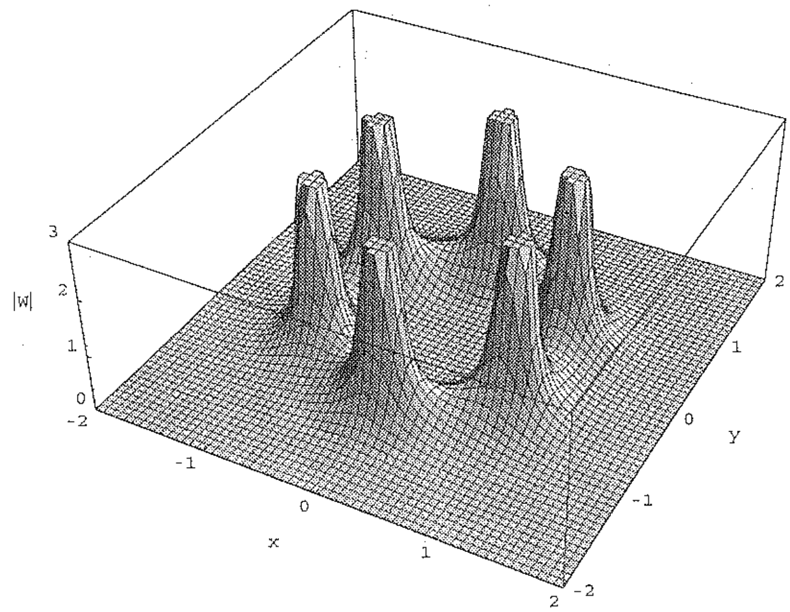

Figure 5. Surface plot of $|W|$ in polar coordinates from eq. (17) for $H_0 = r_0 = 1$ and $\alpha = 6$.

Nielsen–Olesen instability drives the ferromagnetic state to a zero chromomagnetic field state.

The nontrivial vector potentials that we have obtained may be related to confinement problem of QCD just like Gribov's vacuum solutions [14]. The polar plot of our analytical solution for W looks similar to the numerical solution of electroweak condensate of ref. [13]. Flux tube structure of $F^{\mu\nu}$ may correspond to the magnetic field associated with $U(1)$ group and the W field may be a boson condensate.

Modification of vacuum energy of our new solutions due to quantum fluctuations and time evolution of unstable state to our saturated state etc. need to be studied to understand QCD or electroweak vacuum.

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