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String cosmology in inhomogeneous cylindrically symmetric spacetime

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Abstract. String-dust cosmology in an inhomogeneous cylindrically symmetric model is considered. Solutions are obtained only for geometric string with the separability assumption for metric coefficients.

Keywords. String cosmology; inhomogeneous; cylindrical symmetric.

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1. Introduction

The study of cylindrically symmetric spacetime was originated long ago by Lewis [1] in 1932. He solved vacuum Einstein equations in a stationary, cylindrically symmetric model. Later, Van Stockum [2] and Bonnor [3] have done an extensive analysis of the solutions. Recently, Santos [4] has extended the solutions to Einstein equations with a cosmological constant.

In the recent past, Senovilla [5] has shown that inhomogeneous cylindrically symmetric spacetime admits non-singular perfect fluid models satisfying strong energy condition $\rho + 3p > 0$. Here separability of the metric coefficients and the expanding nature of the spacetime are to be assumed. In this paper, we have made an attempt to find solutions of Einstein's equations in a string cosmological model for inhomogeneous spacetime. The motivation for studying such a model is that string theory is a useful concept before creation of the particle in the Universe and has interesting cosmological consequences [6].

The world sheets of the strings are two dimensional time-like surfaces [7]. Here the gravitational effects that arise from strings are considered by the coupling of the stress energy of the strings to the gravitational field for a cylindrically symmetric spacetime. So the energy-momentum tensor for a cloud of massive strings can be written as [8]

$$T_{\mu}^{\gamma} = \rho \cdot V_{\mu} V^{\gamma} - \lambda x_{\mu} x^{\gamma}. \tag{1}$$

Here ρ stands for rest energy for cloud of strings with particles attached to them and λ is string tension density. The connecting relation is

$$\rho = \rho_p + \lambda, \tag{2}$$

with ρ_p as the particle energy density. The four velocity vector V^μ for the cloud of particles and the four vector x^μ, the direction of string will satisfy

$$V_\mu \cdot V^\mu = -1 = -x_\mu x^\mu, \tag{3}$$

and $V_\mu x^\mu = 0$ in +2 signature for the spacetime metric.

2. The basic equations

We consider string cosmology in the general orthogonal cylindrically symmetric spacetime with metric ansatz

$$\mathrm{d}S^2 = D^2\mathrm{d}t^2 - A^2\mathrm{d}r^2 - B^2\mathrm{d}z^2 - c^2\mathrm{d}\,\phi^2, \tag{4}$$

where A, B, C, and D are functions of r and t. The kinematic parameters θ, σ and $\dot{V}_\mu$ are given by

$$\theta = \frac{1}{D}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right), \tag{5}$$

$$\sigma^2 = \frac{1}{9D^2}\left[\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-\frac{2\dot{A}}{A}\right)^2+\left(\frac{\dot{C}}{C}+\frac{\dot{A}}{A}-\frac{2\dot{B}}{B}\right)^2+\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{2\dot{C}}{C}\right)^2\right], \tag{6}$$

$$\dot{V}_\mu = -\frac{\dot{D}}{D}, \tag{7}$$

where '.' and '/' over the letters stand for $\partial/\partial t$ and $\partial/\partial\gamma$ respectively. Now, with the above energy-momentum tensor (see eq. (1)) for string-dust system, the Einstein equations read as

$$R_{\mu\gamma} = -8\pi(\rho V_\mu V_\gamma - \lambda x_\mu x_\gamma - \tfrac{1}{2}(\rho+\lambda)g_{\mu\gamma}), \tag{8}$$

which in the comoving coordinate system with

$$V_\mu = (D^{-1}, 0, 0, 0)$$

and

$$x_\mu = (0, 0, B^{-1}, 0),$$

take the form

$$\frac{1}{D^2}\left[\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}-\frac{\dot{D}}{D}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\right]-\frac{1}{A^2}\left[\frac{D''}{D}+\frac{D'}{D}\left(\frac{B'}{B}+\frac{C'}{C}-\frac{A'}{A}\right)\right] = -4\pi(\rho-\lambda), \tag{9}$$

$$-\frac{1}{A^2}\left[\frac{B''}{B}+\frac{C''}{C}+\frac{D''}{D}-\frac{A'}{A}\left(\frac{B'}{B}+\frac{C'}{C}+\frac{D'}{D}\right)\right]+\frac{1}{D^2}\left[\frac{\ddot{A}}{A}+\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-\frac{\dot{D}}{D}\right)\right] = 4\pi(\rho+\lambda), \tag{10}$$

$$-\frac{1}{A^2}\left[\frac{B''}{B}+\frac{B'}{B}\left(\frac{C'}{C}+\frac{D'}{D}-\frac{A'}{A}\right)\right]+\frac{1}{D^2}\left[\frac{\ddot{B}}{B}+\frac{\dot{B}}{B}\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}-\frac{\dot{D}}{D}\right)\right] = -4\pi(\lambda-\rho), \tag{11}$$

$$-\frac{1}{A^2}\left[\frac{C''}{C}+\frac{C'}{C}\left(\frac{B'}{B}+\frac{D'}{D}-\frac{A'}{A}\right)\right]+\frac{1}{D^2}\left[\frac{\ddot{C}}{C}+\frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{\dot{D}}{D}\right)\right] = 4\pi(\rho+\lambda), \tag{12}$$

$$\frac{1}{AD}\left[\frac{\dot{B}'}{B}+\frac{\dot{C}'}{C}-\frac{\dot{A}}{A}\left(\frac{B'}{B}+\frac{C'}{C}\right)-\frac{D'}{D}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\right]=0. \tag{13}$$

As there are five field equations containing six unknowns, so to get solutions we have to assume an equation of state for the string-dust system.

3. Solutions and their physical implications

If we consider the geometrical string system i.e. the equation of state to be

$$\rho = \lambda \text{ (i.e. } \rho_p = 0)$$

then the above field equations simplify to

$$\frac{1}{D^2}\left[\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}-\frac{\dot{D}}{D}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\right]-\frac{1}{A^2}\left[\frac{D''}{D}+\frac{D'}{D}\left(\frac{B'}{B}+\frac{C'}{C}-\frac{A'}{A}\right)\right]=0, \tag{14}$$

$$-\frac{1}{A^2}\left[\frac{B''}{B}+\frac{C''}{C}+\frac{D''}{D}-\frac{A'}{A}\left(\frac{B'}{B}+\frac{C'}{C}+\frac{D'}{D}\right)\right] + \frac{1}{D^2}\left[\frac{\ddot{A}}{A}+\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-\frac{\dot{D}}{D}\right)\right]=8\pi\lambda, \tag{15}$$

$$-\frac{1}{A^2}\left[\frac{B''}{B}+\frac{B'}{B}\left(\frac{C'}{C}+\frac{D'}{D}-\frac{A'}{A}\right)\right]+\frac{1}{D^2}\left[\frac{\ddot{B}}{B}+\frac{\dot{B}}{B}\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}-\frac{\dot{D}}{D}\right)\right]=0, \tag{16}$$

$$-\frac{1}{A^2}\left[\frac{C''}{C}+\frac{C'}{C}\left(\frac{B'}{B}+\frac{D'}{D}-\frac{A'}{A}\right)\right]+\frac{1}{D^2}\left[\frac{\ddot{C}}{C}+\frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{\dot{D}}{D}\right)\right]=8\pi\lambda, \tag{17}$$

$$\frac{1}{AD}\left[\frac{\dot{B}'}{B}+\frac{\dot{C}'}{C}-\frac{\dot{A}}{A}\left(\frac{B'}{B}+\frac{C'}{C}\right)-\frac{D'}{D}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\right]=0. \tag{18}$$

Now without any loss of generality let us assume that $\dot{D}=0$ and $A'=0$. Further we assume the metric coefficients to be in separable form as follows:

$$A = A_t(t), B(r,t) = B_r(r)\cdot B_t(t), C(r,t) = C_t(t)\cdot C_t(t), D = D_r(r). \tag{19}$$

Then two separate ordinary differential equations obtained from each (14) and (16) are as follows:

$$\frac{\ddot{A}_t}{A_t}+\frac{\ddot{B}_t}{B_t}+\frac{\ddot{C}_t}{C_t}=m_1/A_t^2, \tag{14a}$$

$$\frac{D_r''}{D_r}+\frac{D_r'}{D_r}\left(\frac{B_r'}{B_r}+\frac{C_r'}{C_r}\right)=m_1/D_r^2, \tag{14b}$$

and

$$\frac{\ddot{B}_t}{B_t} + \frac{\dot{B}_t}{B_t}\left(\frac{\dot{A}_t}{A_t} + \frac{\dot{C}_t}{C_t}\right) = m_2/A_t^2, \tag{16a}$$

$$\frac{B''_r}{B_r} + \frac{B'_r}{B_r}\left(\frac{C'_r}{C_r} + \frac{D'_r}{D_r}\right) = m_2/D_r^2, \tag{16b}$$

where m_1 and m_2 are constants of separation.

By subtracting (15) from (17) and then using the separable form (19) we obtain

$$\frac{\ddot{A}_t}{A_t} - \frac{\ddot{C}_t}{C_t} + \frac{\dot{B}_t}{B_t}\left(\frac{\dot{A}_t}{A_t} - \frac{\dot{C}_t}{C_t}\right) = m_3/A_t^2 \tag{15a}$$

and

$$\frac{B''_r}{B_r} + \frac{D''_r}{D_r} - \frac{C'_r}{C_r}\left(\frac{B'_r}{B_r} + \frac{D'_r}{D_r}\right) = m_3/D_r^2, \tag{15b}$$

(m_3 is another constant of separation).

Finally, eq. (18) in separable form becomes

$$\frac{B'_r}{B_r}\cdot\frac{\dot{B}_t}{B_t} + \frac{C'_r}{C_r}\cdot\frac{\dot{C}_t}{C_t} - \frac{\dot{A}_t}{A_t}\cdot\left(\frac{B'_r}{B_r} + \frac{C'_r}{C_r}\right) - \frac{D'_r}{D_r}\left(\frac{\dot{B}_t}{B_t} + \frac{\dot{C}_t}{C_t}\right) = 0. \tag{18a}$$

We shall now determine solutions of the above equations in the following situations:

Step I. When all the separation constants are zero.

From (15a) and (16a) we get first integrals as

$$\frac{\dot{A}_t}{A_t} - \frac{\dot{C}_t}{C_t} = \frac{\alpha_1}{A_t\cdot B_t\cdot C_t}, \tag{20}$$

and

$$\frac{\dot{B}_t}{B_t} = \frac{\alpha_2}{A_t\cdot B_t\cdot C_t}, \tag{21}$$

where α_1 and α_2 are constants of integration.

Similarly from (16b) the first integral gives

$$\frac{B'_r}{B_r} = \frac{\delta_1}{B_r C_r D_r}. \tag{22}$$

Also after some simple algebra with eqs. (14b), (15b) and (16b), we get (integrating once)

$$\frac{D'_r}{D_r} = \delta_2/B_r C_r D_r, \tag{23}$$

and

$$\frac{C'_r}{C_r} = \delta_3/B_r C_r D_r,$$

where δ_1, δ_2 and δ_3 are integration constants. Now, substituting these first integrals in (18a) we get

$$\frac{\dot{A}_t}{A_t} + \frac{\dot{C}_t}{C_t} = \mu/(A_t, B_t C_t).$$

Hence,

$$\frac{\dot{A}_t}{A_t} = \frac{\alpha_3}{A_t B_t C_t}, \frac{\dot{C}_t}{C_t} = \frac{\alpha_4}{A_t B_t C_t}. \tag{24}$$

Solving these first integrals further using the above field equations we have

$$\left.\begin{array}{l} A_t \, \alpha \, t^{\mu_1} \\ B_t \, \alpha \, t^{\mu_2} \\ C_t \, \alpha \, t^{\mu_3} \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} D_r \, \alpha \, r^{\gamma_1} \\ B_r \, \alpha \, r^{\gamma_2} \\ C_r \, \alpha \, r^{\gamma_3} \end{array}\right\} \tag{25}$$

where the constants (μ_1, μ_2, μ_3) and $(\gamma_1, \gamma_2, \gamma_3)$ which satisfy

$$\mu_1 + \mu_2 + \mu_3 = 1 = \mu_1^2 + \mu_2^2 + \mu_3^2,$$

and

$$\gamma_1 + \gamma_2 + \gamma_3 = 1 = \gamma_1^2 + \gamma_2^1 + \gamma_3^2. \tag{26}$$

Also the inter-relation between the two sets of parameters is

$$\mu_1\gamma_1 + \mu_2\gamma_2 + \mu_3\gamma_1 + (1 - \mu_1)(1 - \gamma_1) = 1. \tag{27}$$

So the above solution can be termed as an inhomogeneous Kasner solution. In this case the string tension vanishes i.e. there is no string concept when all the separation constants are assumed to be zero.

Step II. Let us assume $D = 1$.

Then from (14) we get

$$\frac{\ddot{A}_t}{A_t} + \frac{\ddot{B}_t}{B_t} + \frac{\ddot{C}_t}{C_t} = 0. \tag{28}$$

The separable form for (18) and (16) are

$$\frac{C_r'/C_r}{B_r'/B_r} = \frac{\dot{B}_t/B_t - \dot{A}_t/A_t}{\dot{A}_t/A_t - \dot{C}_t/C_t} = m_4, \tag{29}$$

and

$$\frac{B_r''}{B_r} + \frac{B_r'}{B_r} \cdot \frac{C_r'}{C_r} = m_5, \tag{30}$$

$$\frac{\ddot{B}_t}{B_t} + \frac{\dot{B}_t}{B_t}, \left(\frac{\dot{A}_t}{A_t} + \frac{\dot{C}_t}{C_t}\right) = m_5/A_t^2. \tag{31}$$

Also the separable form for the resulting equation after subtracting eq (15) from (17) is

$$\frac{\ddot{A}_t}{A_t}-\frac{\ddot{C}_t}{C_t}+\frac{\dot{B}_t}{B_t},\left(\frac{\dot{A}_t}{A_t}-\frac{\dot{C}_t}{C_t}\right)=m_5/A_t^2. \tag{32}$$

(The equation for space part is the same as in eq. (30)).

Now, eliminating C_r between eq. (29) and (30), we have

$$\frac{B_r''}{B_r}+m_4\frac{B_r'^2}{B_r^2}=m_5 \tag{33}$$

which has a first integral as

$$B_r'^2=\frac{m_5}{1+m_4}B_r^2+C\cdot B_r^{-2m_4}, \quad (m_4\neq -1) \tag{34}$$

with C as the constant of integration. The integral form of the above equation is

$$\pm(r-r_0)=\int\frac{\mathrm{d}B_r\cdot B_r^{m_4}}{\sqrt{\frac{m_5}{1+m_4}B_r^{2(1+m_4)}+C}}, \tag{35}$$

where r_0 is another integration constant.

We shall solve the above integral for different cases as follows:

Case I. When $m_4=-1$.

In this case eq. (33) can be integrated easily to give

$$B_r=\exp\left[\frac{m_5(r-r_0)^2}{2}-\frac{C}{2m_5}\right], \tag{36}$$

and

$$C_r=\exp\left[\frac{C}{2m_5}-\frac{m_5}{2}(r-r_0)^2\right].$$

For the time part we have

$$\frac{\dot{B}_t}{B_t}=\frac{\dot{C}_t}{C_t} \quad \text{for } m_4=-1. \tag{37}$$

The solution is

$$A_t\alpha t^{-1/3}, \quad B_t\alpha t^{2/3}, \quad C_t\alpha t^{2/3}. \tag{38}$$

So the solution is again in Kasner type for time variable only, the space part is in the exponential form. Hence we can call it a semi-inhomogeneous Kasner type solution.

In this case the expression for string tension and other kinematical parameters are

$$\lambda=\mu t^{2/3}\cdot(r-r_0)^2-\gamma/t^2,$$

$$\theta\alpha\frac{1}{t},$$

$$\sigma^2\alpha\frac{1}{t^2},$$

where μ and γ are constants depending on the parameters involved in the solution. So string tension is not homogeneous, it depends on both space and time coordinates. The peculiarity of the expression for λ is that it starts from $-\infty$ at $t = 0$ and goes to $+\infty$ at $t = \infty$. Hence string will remain even after infinite time. Also both the expansion scalar and shear scalar starts with infinite value at $t = 0$ and then gradually decreases and finally the expansion halts and the Universe becomes isotropic at $t = +\infty$.

Case II. When $m_4 = 0$.

The explicit expression for the solutions are

$$B_r = \begin{cases} \sqrt{\frac{C}{m_5}}\,\mathrm{Sinh}[\sqrt{m_5}(r - r_0)], & C > 0, \\ \sqrt{\frac{|C|}{m_5}}\,\mathrm{Cosh}[\sqrt{m_5}(r - r_0)], & C < 0, \end{cases} \tag{40}$$

$$C_r = \text{Constant}.$$

For the time part all the scale factors are linearly proportional to time. Thus the expression for γ, θ and σ are

$$\begin{aligned} \lambda &= \lambda_0/t^2, \\ \theta &= 3/t, \\ \sigma^2 &= 0, \end{aligned} \tag{41}$$

where $\lambda_0 = (2 - m_5/a_0^2)/8\pi$ with $A_t = a_o \cdot t$. So in this case, the string tension has infinite value at the initial epoch and then gradually diminishes with the evolution of the Universe and finally string will disappear after infinite time. It is interesting to note that though the metric coefficients are inhomogeneous yet string tension depends purely on time.

Moreover, this solution corresponds to isotropic Universe and the expansion scalar diminishes with time and finally vanishes. Also from the string energy condition $\lambda_0 \le 0$ i.e. $m_5 \ge 2a_0^2$, is the restriction involved in the solution.

Finally, it is to be noted that the solution (40) (for $C < 0$) is similar to that of Tikekar *et al* [9] for $b = 1$. But the two solutions are not identical because we have considered geometric string while they have considered barotropic equation of state.

Further, it is to be noted that the integral (35) can be solvable for $m_4 = -1/2$ and $m_4 = 1$ also. The results are

$$B_r = \begin{cases} \frac{C}{2m_5}\,\mathrm{Sinh}^2[\sqrt{m_5/2}(r - r_0)] & \text{for } C > 0, \\ \frac{|C|}{2m_5}\,\mathrm{Cosh}^2[\sqrt{m_5/2}(r - r_0)] & \text{for } C < 0, \end{cases}$$

$$C_r = \begin{cases} \sqrt{2m_5/C}\,\mathrm{Cosech}[\sqrt{m_5/2}(r - r_0)] & \text{for } C > 0, \\ \sqrt{2m_5/|C|}\,\mathrm{Sech}[\sqrt{m_5/2}(r - r_0)] & \text{for } C < 0, \end{cases}$$

for $m_4 = -1/2$, and

$$B_r^2 = C_r^2 = \begin{cases} \sqrt{\frac{2C}{m_5}}\,\mathrm{Sinh}[\sqrt{2m_5}(r - r_0)] & \text{for } C > 0, \\ \sqrt{\frac{2|C|}{m_5}}\,\mathrm{Cosh}[\sqrt{2m_5}(r - r_0)] & \text{for } C < 0, \end{cases}$$

for $m_4 = 1$.

But it is not possible to find explicit solutions for the time part.

Finally, we note that due to the presence of the string concept the metric cannot be obtained as singularity free, only space part is singularity free in one case. Therefore, for future work it will be interesting to study in detail whether singularity free Senovilla type solutions are possible or not.

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