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Non-static local string in Brans–Dicke theory

A A SEN*
Relativity and Cosmology Research Centre, Department of Physics, Jadavpur University, Calcutta 700 032, India
*Present Address: Mehta Research Institute, Chattnag Road, Jhusi, Allahabad 211 019, India
Email: anjan@mri.ernet.in

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Abstract. A recent investigation showed that a local gauge string with a phenomenological energy momentum tensor, as prescribed by Vilenkin, is inconsistent in Brans–Dicke theory. In this work it has been shown that such a string is indeed consistent if one introduces time dependences in the metric. A set of solutions of full non-linear Einstein's equations for the interior region of such a string are presented.

Keywords. Strings; B–D theory.

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The importance of non-minimally coupled scalar tensor theories such as Brans–Dicke theory, in the context of inflationary scenario has generated considerable interest in the gravitational field of certain topological defects, particularly cosmic strings [1–6]. Very recently it has been shown by the present authors that a local static gauge string, given by the energy momentum tensor components $T_t^t = T_z^z \neq 0$ and all other $T_\nu^\mu = 0$ [7] is inconsistent in B–D theory of gravity [8], although it is quite consistent in a more general scalar tensor theory [9], where the parameter ω is a function of the B–D scalar field.

In this work, we have shown that one can obtain a consistent set of non-static solutions for full non-linear gravitational field equations for local gauge string with T_ν^μ as mentioned above in B–D theory. Very recently Dando and Gregory [10] also obtained non-static solutions for a global string in dilaton gravity.

The gravitational field equations in B–D theory are given by

$$G_{\mu\nu} = \frac{T_{\mu\nu}}{\phi} + \frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi^{,\alpha}\phi_{,\alpha}\right) + \frac{1}{\phi}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi), \tag{1}$$

where ω is a dimensionless constant parameter of the theory and $\phi = \phi(r)$ is the B–D scalar field.

The wave equation for the scalar field ϕ is

$$\Box\phi = \frac{T}{(2\omega + 3)}. \tag{2}$$

In these equations, $T_{\mu\nu}$ represents the energy momentum tensor components for all the fields except the scalar field ϕ and T is the trace of $T_{\mu\nu}$.

We take the line element describing a cylindrically symmetric spacetime, with two killing vectors ∂_z and ∂_ϕ as in [11],

$$\mathrm{d}s^2 = \mathrm{e}^{2A(r)}[\mathrm{d}t^2 - \mathrm{e}^{2B(t)}\mathrm{d}z^2] - D^2(t)[\mathrm{d}r^2 + C^2(r)\mathrm{d}\theta^2], \tag{3}$$

where r, θ, z are cylindrical coordinates defined in the range $0 \leq r \leq \infty; 0 \leq \theta \leq 2\pi - \infty \leq z \leq \infty$.

The local gauge string is characterized by an energy density and a stress along the symmetry axis, given by [7]

$$T_t^t = T_z^z = \sigma(r), \tag{4a}$$

$$T_r^r = T_\theta^\theta = 0. \tag{4b}$$

Similar kind of energy momentum tensor has been considered by Barros and Romero [2] in their investigation of local gauge string in B–D theory in a linearized approximation of the field equations.

The conservation of matter is represented by the equation

$$T^{\mu\nu}_{;\nu} = 0. \tag{5}$$

It should be noted that (5) and (2) are not independent equations, as in view of (1) and the Bianchi identity, one yields the other.

Equation (5) together with (4a), (4b) and (3) yields

$$\dot{D} = 0, \tag{6a}$$

$$A' = 0 \tag{6b}$$

for $\sigma \neq 0$. Here an overhead dot and prime represent differentiations with respect to t and r respectively.

So D and A are constants and in what follows we shall take $D = 1$ and $\mathrm{e}^{2A} = 1$ which leads only to a rescaling of coordinates and no loss of generality.

In view of (6a) and (6b), the field equations (1) can be written as

$$\frac{C''}{C} = -\frac{\sigma}{\phi} - \frac{\omega}{2}\frac{\phi'^2}{\phi^2} - \frac{\phi''}{\phi} - \frac{\phi'}{\phi}\frac{C'}{C}, \tag{7a}$$

$$\ddot{B} + \dot{B}^2 = -\frac{\omega}{2}\frac{\phi'^2}{\phi^2} + \frac{\phi'}{\phi}\frac{C'}{C}, \tag{7b}$$

$$\ddot{B} + \dot{B}^2 = \frac{\omega}{2}\frac{\phi'^2}{\phi^2} + \frac{\phi''}{\phi}. \tag{7c}$$

The wave equation (2) can be written as

$$\frac{\phi''}{\phi} + \frac{\phi'}{\phi}\frac{C'}{C} = -\frac{2\sigma}{(2\omega+3)\phi}. \tag{7d}$$

From (7b) and (7c) one can write

$$\ddot{B} + \dot{B}^2 = -\frac{\omega}{2}\frac{\phi'^2}{\phi^2} + \frac{\phi'}{\phi}\frac{C'}{C} = \frac{\omega}{2}\frac{\phi'^2}{\phi^2} + \frac{\phi''}{\phi} = b_0, \tag{8}$$

where b_0 is a separation constant.

From (8) one gets

$$\frac{\phi''}{\phi} + \frac{\phi'}{\phi}\frac{C'}{C} = 2b_0. \tag{9}$$

Hence the equation to be solved are

$$\frac{C''}{C} = -\frac{\sigma}{\phi} - \frac{\omega}{2}\frac{\phi'^2}{\phi^2} - 2b_0, \tag{10a}$$

$$\frac{\phi'}{\phi}\frac{C'}{C} - \frac{\omega}{2}\frac{\phi'^2}{\phi^2} = b_0, \tag{10b}$$

$$\frac{\phi''}{\phi} + \frac{\phi'}{\phi}\frac{C'}{C} = 2b_0, \tag{10c}$$

$$\frac{\sigma}{(2\omega+3)\phi} = -b_0, \tag{10d}$$

$$\ddot{B} + \dot{B}^2 = b_0. \tag{10e}$$

Here we have four unknowns C, ϕ, σ, B and we have five equations. However one can show that (10a) is not an independent equation but can be obtained from the rest of the equations.

In what follows we shall try to solve the system of equations for $b_0 = 0$ and $b_0 \neq 0$.

Case I: $b_0 = 0$

In this case, from (10e), one can solve B which becomes

$$\mathrm{e}^B = t, \tag{11}$$

where the constants of integration have been absorbed by rescaling the time coordinate without any loss of generality.

A coordinate transformation of the form [12] $\xi = t\sinh z$ and $\tau = t\cosh z$ reduces this solution to the static form. That is the case $b_0 = 0$ essentially represents a static case.

From (10d), however, one can see that either $\sigma = 0$ for finite value of ω, or $\omega \to \infty$ for finite value of σ. But we know that for large value of ω, the B–D theory is indistinguishable from GR when $T = T^\mu_\mu \neq 0$ although for $T = T^\mu_\mu = 0$ it is not so [13]. Hence we cannot have a cosmic string with T^μ_ν given by (4a) and (4b) in B–D theory for $b_0 = 0$ which is a static case. This result is completely in agreement with the previous work [8] that a static local gauge string of Vilenkin type does not exist in B–D theory of gravity.

Case II: $b_0 \neq 0$

In this case, one should note from eq. (10d) that if $\phi > 0$, which ensures the positivity of G, the gravitational constant, and $(2\omega + 3)$ is also positive, then to have positive energy density σ, b_0 should be negative and hence we take

$$b_0 = -b_1^2, \tag{12}$$

where b_1 is a real constant.

It deserves mention at this point that Gregory [14] found a non-singular solution for a global string in GR for non-static case. For that, one has to take a positive value of the constant b_0, and even for the global string, the energy density corresponding to the string (T^0_0, to be precise) takes a negative value for a certain value of the radial distance from the axis of the string.

Now from (10e), we have

$$\frac{(\mathrm{e}^{\ddot{B}})}{\mathrm{e}^B} = -b_1^2$$

which on integration yields

$$\mathrm{e}^B = M\sin(b_1 t) + N\cos(b_1 t), \tag{13}$$

where M and N are constants of integration.

From (10b) and (10c) one gets,

$$\frac{\phi''}{\phi} + \frac{\omega}{2}\frac{\phi'^2}{\phi^2} = -b_1^2. \tag{14}$$

Solving eq. (14), one gets

$$\phi = \phi_0[\cos(b_1 ar)]^{1/a^2}, \tag{15}$$

where $a^2 = (\omega + 2)/2$ and ϕ_0 is a constant of integration.

Again from (10b) and (10c) one gets

$$\frac{\phi''}{\phi} + \omega\frac{\phi'^2}{\phi^2} = \frac{\phi'}{\phi}\frac{C'}{C}. \tag{16}$$

Now if one assumes $(\phi'/\phi) \neq 0$, then one can integrate the above equation to yield

$$C = K\phi'\phi^\omega, \tag{17}$$

where K is a constant of integration.

Using (15) and (16) one gets,

$$C^2 = C_0^2 \sin^2(b_1 ar)[\cos(b_1 ar)]^{2\omega/(\omega+2)}, \tag{18}$$

where $C_0^2 = (2b_1^2 K^2 \phi_0^{2(\omega+1)})/(\omega+2)$.

The string energy density σ can be found from (10d) which becomes

$$\sigma = \phi_0 b_1^2 (2\omega+3)[\cos(b_1 ar)]^{2/(\omega+2)}. \tag{19}$$

Finally the line element becomes

$$\begin{aligned} \mathrm{d}s^2 = {} & \mathrm{d}t^2 - \mathrm{d}r^2 - [M\sin(b_1 t) + N\cos(b_1 t)]^2 \mathrm{d}z^2 - C_0^2 \sin^2(b_1 ar) \\ & \times [\cos(b_1 ar)]^{2\omega/(\omega+2)} \mathrm{d}\theta^2. \end{aligned} \tag{20}$$

One should note that for $r \to 0$ i.e. near the axis of the string, the line element becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - [M\sin(b_1 t) + N\cos(b_1 t)]^2 \mathrm{d}z^2 - C_0^2 b_1^2 a^2 r^2 \mathrm{d}\theta^2. \tag{21}$$

The non-static metric (20) shows that the time dependence is only along the symmetry axis where the constant $b_0 = -b_1^2$ acts like an effective cosmological constant along the symmetry axis. The case is similar to that obtained earlier by Gregory [14] for the case of a non-static global string in GR and also by Dando and Gregory [10] for dilatonic global string.

One can calculate the scalar curvature for the spacetime

$$R = 2\omega b_1^2 \left[\frac{\tan^2(b_1 ar)}{(\omega+2)} - \frac{2}{(2\omega+3)} \right] \tag{22}$$

which is evidently time independent. But one can check that the spacetime becomes singular periodically with r which is due to the form of the temporal behaviour along the symmetry axis. Because the periodic time dependence along z-axis is due to $b_0 < 0$ and because of this negative b_0, the solution of the radial part is such that one gets a $\tan^2(b_1 ar)$ function in the expression for the scalar curvature. But for positive b_0 this function would be replaced by $\tanh^2(b_1 ar)$ and there would be no singularity in the spacetime. Hence although there is no explicit time dependence in the expression for the scalar curvature but it is the temporal behaviour of the spacetime that determines the singularity nature for the spacetime. It is similar to that obtained by Gregory for the non-static global string [14]. There also for periodic time dependence along the symmetry axis, spacetime has physical singularity but for exponential time dependence the spacetime is non-singular. In our case we have to choose negative b_0 to ensure the positivity of the energy density and that is why the spacetime is necessarily singular.

To conclude, we have shown that although static local string prescribed by Vilenkin is incompatible with the Brans–Dicke theory of gravity one can have exact solution for the gravitational field equations for such a non-static string in Brans–Dicke theory. The spacetime has an oscillatory time dependence along the symmetry axis which in turn causes the occurrence of a periodic physical singularity at different finite distances from the symmetry axis. This periodic time dependence is due to ensure the positivity of the energy density of the string.

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