PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 55, No. 4
October 2000
pp. 623–628

On spherically symmetric singularity-free models in relativistic cosmology

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Abstract. The introduction of time dependence through a scale factor in a non-conformally flat static cosmological model whose spacetime can be embedded in a five demensional flat spacetime is shown to give rise to two spherical models of universe filled with perfect fluid acompannied with radial heat flux without any Big Bang type singularity. The first model describes an ever existing universe which witnesses a transition from state of contraction to that of ever expansion. The second model represents a universe oscillating between two regular states.

The discovery of non-singular relativistic cosmological models by Senovilla [1] describing cylindrically symmetric universes filled with a perfect fluid provided impetus for exploring the possibilities of constructing spherically symmetric non-singular cosmological models in the context of general relativity. Dadhich *et al* [2,3] reported a couple of models one describing a non-singular spherically symmetric universes devoid of any Big Bang type singularity, filled with non-adiabatic fluid with anisotropic pressure accompanied with heat flux along radial direction, while the other with a perfect fluid source in the presence of null radiation flowing along radial directions. An interesting feature of the spacetime metric of these models is that it contains an arbitrary function of time which can be constrained to comply with the demands of non-singularity and energy conditions. Dadhich and Raychaudhuri [4] later showed how a particular choice of this function leads to a model of an ever existing spherically symmetric universe, oscillating between two regular states, which involves blue shifts as in the quasi steady state cosmological model of Hoyle, Burbige and Narlikar [5] and is filled with a non-adiabatic fluid with anisotropic pressure and radial heat flux. These observations led to the search of spherically symmetric singularity-free cosmological models with a perfect fluid source characterized by isotropic pressure This search resulted in construction of two spherically symmetric singularity-free relativistic cosmological models, describing universes filled with non-adiabatic perfect fluid accompanied by heat flow along radial direction which are reported in this talk. The first model describes an ever existing universe witnessing a phase of contraction that is subsequently followed by an everlasting phase of expansion. The second model corresponds to an everexisting universe oscillating between two regular states. As in QSSC, this latter model will also predict blueshifts, should they be observed in future, but without invoking non-conservation of energy or violation of GR, with a non-adiabatic perfect fluid source accompanied with heat flux, which seems to be a remarkable and interesting feature of this model.

Spacetime of this singularity-free cosmological model is described by metric

$$\mathrm{d}s^2 = (1+\alpha r^2)\mathrm{d}t^2 - g(t)\left(\frac{1+2\alpha r^2}{1+\alpha r^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right), \quad (1)$$

where $\alpha > 0$, is a constant.

If $g(t) = 1$, this spacetime metric has the interesting feature [6], that it describes a static cosmological solution of EFEs with a perfect fluid source which is not conformally flat and can be embedded in a five dimensional flat spacetime. It should be noted here that the spacetime of the FRW-model of the universe can be embedded in a five dimensional flat spacetime but has geometry conformal to a flat spacetime.

The physical content of the spacetime of these models is stipulated to be non-adiabatic perfect fluid accompanied with heat flux along radial direction with the energy-momentum tensor

$$T_i^j = (\rho + p)u_i u^j - p\delta_i^j + q_i u^j + u_i q^j, u^i u_i = 1 \quad (2)$$

and the co-ordinates are chosen to follow the motion of the fluid so that the four velocity u^i and the heat flux vector q^i take on the respective expressions

$$u^i = \left(0, 0, 0, \frac{1}{\sqrt{1+\alpha r^2}}\right), \quad (3)$$

$$q^i = (q, 0, 0, 0). \quad (4)$$

Einstein's field equations relate the dynamical variables ρ, p and q corresponding to the matter density, fluid pressure and heat flux parameter with the metric potentials and imply

$$8\pi\rho = \frac{\alpha(3+2\alpha r^2)}{g(1+2\alpha r^2)^2} + \frac{3}{4(1+\alpha r^2)}\frac{\dot{g}^2}{g^2}, \quad (5)$$

$$8\pi p = \frac{\alpha}{g(1+2\alpha r^2)} - \frac{1}{(1+\alpha r^2)} \cdot \left(\frac{\ddot{g}}{g} - \frac{1}{4}\frac{\dot{g}^2}{g^2}\right), \quad (6)$$

$$8\pi q = \frac{-\alpha r}{(1+2\alpha r^2)(1+\alpha r^2)} \cdot \frac{\dot{g}}{g^2}. \quad (7)$$

The metric function $g(t)$ will be chosen so that the metric and the above dynamical variables remain finite everywhere all the time and the respective model is non-singular.

Model 1

The spacetime metric of the model-1 follows on setting: $g(t) = (at^2 + b), a > 0, b > 0$.

It represents a universe with the dynamical variables ρ, p and q of its matter content having the explicit expressions

$$8\pi\rho = \frac{\alpha(3+2\alpha r^2)}{(at^2+b)(1+2\alpha r^2)^2} + \frac{3a^2t^2}{(at^2+b)^2(1+\alpha r^2)}, \tag{8}$$

$$8\pi p = \frac{(\alpha-2a)+\alpha(\alpha-4a)r^2}{(1+\alpha r^2)(1+2\alpha r^2)(at^2+b)} + \frac{a^2t^2}{(1+\alpha r^2)(at^2+b)^2}, \tag{9}$$

$$8\pi q = \frac{-2\alpha r}{(1+\alpha r^2)(1+2\alpha r^2)} \cdot \frac{at}{(at^2+b)^2}. \tag{10}$$

It is evident that the matter density is always and everywhere positive while positivity of pressure is ensured if $\alpha \geq 4a$. The heat flux parameter $q > 0$ for $t < 0,\ q = 0$ for $t = 0$ and $q < 0$ for $t > 0$.

The expression for kinematic parameter of expansion θ reads

$$\theta = \frac{3}{\sqrt{(1+\alpha r^2)}} \frac{at}{(at^2+b)} \tag{11}$$

and implies that the model describes a contracting universe for $t < 0$ with $q > 0$ and an expanding universe for $t > 0$ with $q < 0$, the switching from contracting phase to phase of expansion occurring at $t = 0$.

From the expressions for matter density and pressure it follows that $p < \rho$, while the positivity of

$$8\pi(\rho - 3p) = \frac{2}{(at^2+b)} \left\{ \frac{3a + 2\alpha(6a-\alpha)r^2 + 2\alpha^2(6a-\alpha)r^4}{(1+2\alpha r^2)(1+\alpha r^2)} \right\}, \tag{12}$$

everywhere is ensured if $\alpha \leq 6a$. The positivity of pressure demands $\alpha \geq 4a$. If the pressure is positive i.e., $\alpha \geq 4a$ it follows that

$$(\rho + p)^2 - 4q^2 \geq 0.$$

Accordingly we restrict α to comply with

$$4a \leq \alpha \leq 6a, \tag{13}$$

so that the requirements of all (weak, strong and dominant) energy conditions are fulfilled.

If T denotes the temperature distribution of the fluid, the heat flux vector q^i is related with it through the phenomenological heat conduction equation

$$q^i = -K(g^{ij} - u^i u^i)(T_{,j} + T u_{j,k} u^k), \tag{14}$$

the equation which ensures positivity of entropy flux production with K denoting the conductivity which is expected to be proportional to temperature.

If we stipulate the dependence of K on T in the universe of the model under consideration as to follow, from

$$K = \gamma T^{\Omega+1} = \gamma F, \tag{15}$$

where $\Omega > 0$ is a constant.

The phenomenological conduction equation integrates out to give

$$T^{\Omega+1} = \beta(\alpha r^2 + 1)^{\frac{\Omega+1}{2}} + \frac{2(\Omega+1)}{\gamma(\Omega+3)} \frac{at}{(at^2+b)(\alpha r^2+1)}, \tag{16}$$

where β is the arbitrary constant of integration. If it is chosen as zero, the finite temperature is measured everywhere for all time.

The overall evolution of the universe of this model can be described as follows. The universe begins to contract from a state of infinite dilution with vanishingly small matter density at infinitely remote past and remains in the contracting phase during the period $(-\infty < t < 0)$. The switching from the phase of contraction to that of expansion occurs at $t = 0$, the epoch when matter density at the centre has reached its maximum value $\rho_{\max} = 3\alpha/b$, which can be made as large as desired by choosing α and b. The universe then expands forever to reach the state of infinite dilution at infinitely remote future, ρ, p and T remaining regular everywhere all the time. Thus the model-1 describes a singularity-free universe which exists all the time, contracting for $t < 0$ and there after expanding forever.

Model 2

The second model describing an oscillatory universe is obtained by choosing $g(t) = (a + b\cos\omega t)$ with $a > b > 0$.

The matter density ρ, the fluid pressure p and the heat flux parameter for the non-adiabatic matter content accompanying it are found to have the following respective expressions:

$$8\pi\rho = \frac{\alpha(3+2\alpha r^2)}{(a+b\cos\omega t)(1+2\alpha r^2)^2} + \frac{3}{4}\cdot\frac{(b^2 w^2 \sin^2\omega t)}{(a+b\cos wt)^2(1+\alpha r^2)}, \tag{17}$$

$$8\pi p = \frac{\alpha}{(a+b\cos wt)(1+2\alpha r^2)} + \frac{bw^2(b+4a\cos wt+3b\cos^2 wt)}{4(a+b\cos wt)^2(1+\alpha r^2)}, \tag{18}$$

$$8\pi q = \frac{\alpha r}{(1+\alpha r^2)(1+2\alpha r^2)}\cdot\frac{b\omega t}{at^2+b}. \tag{19}$$

From the expression (18), it follows that the pressure is non-negative if $\alpha \geq 2bw^2$. However the expression

$$8\pi(\rho - 3p) = \frac{4\alpha^2 r^2}{(a+b\cos wt)(1+2\alpha r^2)^w} - \frac{3bw^2\cos wt}{(a+b\cos wt)(1+\alpha r^2)}, \tag{20}$$

implies that the requirement $\rho - 3p > 0$, is not ensured for all time epoch. The second term in

$$8\pi(\rho - p) = \frac{2\alpha}{(a+b\cos wt)(1+2\alpha r^2)^2} + \frac{bw^2(b-3b\cos^2 wt-2a\cos wt)}{2(a+b\cos wt)^2(1+\alpha r^2)} \tag{21}$$

is observed to contribute negatively for $0 \leq \omega t \leq \psi$, where ψ lies between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. We have checked the positivity of $(\rho - p)$ at $\omega t = 0$ observing that

$$8\pi(\rho - p)_{t=0} = \frac{2\alpha}{(a+b)((1+2\alpha r^2)^2} - \frac{bw^2}{(a+b)(1+\alpha r^2)}. \tag{22}$$

If we choose bw^2 to be very small, then for small r correct up to terms of order r^2, positivity of $\rho - p$ follows for $\alpha > 2bw^2$, while $(\rho - p) \longrightarrow 0$ as $r \longrightarrow \infty$. From the expression

$$8\pi(\rho + p) = \frac{4\alpha(1+\alpha r^2)}{(a + b\cos wt)(1+2\alpha r^2)^2} + \frac{bw^2(b + a\cos wt)}{(a+b\cos wt)^2(1+\alpha r^2)}, \tag{23}$$

for $4\alpha > bw^2$ it follows that $(\rho + p)^2 \geq 4q^2$ implying that

$$(\rho - 3p) + [(\rho + p)^2 - 4q^2]^{1/2} > 0.$$

Thus it is observed that the requirements of strong and dominant energy conditions are fulfilled throughout the spacetime of this model.

The expression for the expansion parameter

$$\theta = -\frac{3b\omega \sin \omega t}{2\sqrt{1+\alpha r^2}(a + b\cos \omega t)} \tag{24}$$

indicates that the universe of this model is in the phase of contraction for $2n\pi < \omega t < (2n+1)\pi$ and in the phase of expansion $(2n+1)\pi < \omega t < 2(n+1)\pi$ where n takes on integer values only. During the phase of contraction $q < 0$ while during the expansion phase $q > 0$ with q vanishing when switching from contraction to expansion or vice a versa occurs. The integration of phenomenological equation for conduction of heat as indicated in the discussion of the model I in this case also leads to similar conclusions about the regularity of the temperature throughout the spacetime.

The overall behaviour of the universe of this model may be described as follows. The universe oscillates between two regular states with oscillation period $t = 2\pi/\omega$, with the central matter density attaining its maximum value $\rho_{\max} = 3\alpha/8\pi(a-b)$ at $t = (2n+1)\pi/\omega$ and its minimum value $\rho_{\min} = 3\alpha/8\pi(a+b)$, at $t = 2(n+1)\pi/\omega$. The model can permit as low and as high values for the central matter density as desired for appropriate choice of the parameters a, b. The model involves three parameters a, b and ω, which are related with the maximum and minimum values of matter density at the centre and the duration of an oscillation cycle. The model further being centro symmetric accommodates additional parameters r_0, t_0 corresponding to the radial coordinate and time of observation for the observer. Thus the model has a number of free parameters which can be suitably chosen to bring it closer to observations.

The most interesting feature of the oscillatory model is that the universe according to it, like that of the steady state cosmology has no beginning and no end. Accordingly as in the quasi steady state cosmology, this model would also lead to prediction of blue shifts. The oscillatory model here arises strictly within the framework of general relativity without violating the conservation of matter energy and positivity of energy. Unlike the oscillatory model of Dadhich and Raychaudhuri the universe of this model is filled with a perfect fluid characterized by pressure isotropy. The apparent simplicity of the spacetime metric is indicative of geodesic completeness and causal stability. The model therefore opens up new avenues for further studies which may bear promise for observational cosmology.

Acknowledgements

I thank N Dadhich in collaboration with whom the work reported in this talk was carried out at IUCAA. My thanks are also due to V O Thomas for helpful discussions.

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