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Linear amplifier via the displaced Fock states superposition

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Abstract. The linear amplifier with the superposition of displaced Fock states (DFS's) as an input field is discussed. The s-parameterized characteristic function (CF) of linear amplifier for the superposition of two DFS's is considered. Several quantum statistical expectation values for the output of linear amplifier are evaluated once the time dependent CF has been computed. The Glauber second-order coherence function is calculated. The squeezing properties of the output field are studied. The s-ordered quasiprobability distribution function (QDF) for the output of linear amplifier driven by DFS's superposition is investigated. The phase properties of the superposition of DFS's are studied. The s-parameterized phase distribution, obtained by integrating the s-parameterized QDF over radial variable is illustrated.

Keywords. Nonclassical field states; Fock state; coherent; displaced number; antibunched; sub-Poissonian states; Schrödinger cat states; phase distributions; phase measurements; linear amplifier.

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1. Introduction

The concept of the photon in the quantum theory of a radiation field has been built on the Fock (number) state $|n\rangle$. However, the coherent state is another important state, it may be defined by the action of a displacement operator $D(\alpha)$ on the vacuum state. These states have been extensively studied [1]. On the other hand, the displaced Fock states (DFS's) are very important kinds of states in quantum optics, defined by the action of the displacement operator on the number state [2–4]. They can be regarded as a generalized class of the Fock and coherent states. They form a complete basis, and have interesting and unusual physical properties [2–4]. The quasiprobability distribution functions (QDF's) have been represented as a series in terms of these states [4]. Experiments have been performed to prepare the Fock states, coherent states, and states derived from them, in recent times [5,6]. The various schemes proposed have been built on the motional dynamics of the centre of mass of trapped ions [5].

The generation of nonclassical states of light is at the heart of quantum optics. In particular, the superpositions of quantum states [7,8], are considered as an important type of nonclassical states. These states are of particular interest because they possess various nonclassical properties, such as squeezing and sub-Poissonian statistics [8]. Non-classical properties of a superposition of DFS's have been discussed [9]. Particular interest has been devoted to the generation of these states in ref. [9].

The QDF's have already become customary tools of analysing experimental results in detecting quantum states of systems like an ion oscillating in a harmonic trap, or for a mode oscillator [10]. The quantum mechanical systems can be described by QDF's, and quantum mechanical expectation values of operators are calculated completely. The different QDF's are associated with the different orderings of operators. These are not in general true probability densities and they sometimes become negative or more singular than a delta function. In this sense, they are also called quasiprobability densities [11].

It is well known that a linear amplifier modifies the statistical properties of the light amplified. In particular, when the incoming field exhibits sub-Poissonian photon statistics or squeezing, these features can then be lost after amplification when the gain is too high [12–16]. The increased use of lasers in ultra-high precision measurements, especially the work on gravitational wave detection, has focussed the attention on optimization criteria for laser amplifiers. Also, in communication applications the ultimate performance limit may constitute an important factor in dimensioning the optical systems [14].

The purpose of this article is to examine the statistical properties and phase distribution in full generality, by using the DFS's superposition as an initial input field for the linear amplifier. The use of such non-classical states not only lead us to a deeper understanding of the nature of light, but also have applicability to detect weak signals and quantum communications.

This paper is organized as follows. In §2, the construction and properties of superposition of DFS's will be discussed. In §3, the s-parameterized CF on a linear amplifier with DFS's as an input field will be calculated. Some applications for the s-ordered CF: namely; moments and squeezing will be studied. In §4, the s-parameterized QDF will be covered. In §5, the s-parameterized phase distribution will be investigated. Finally, the conclusions are made in §6.

2. Superpositions of displaced Fock states

We start by reviewing some of the properties of DFS's [2–5]. In addition to this, we discuss the properties of the superpositions of these states [9]. The DFS, $|\alpha, n>$, is defined by

$$|\alpha, n\rangle = D(\alpha)|n\rangle \tag{2.1}$$

with $D(\alpha)$ the displacement operator, given by [1]

$$D(\alpha) = \exp(\alpha a^{+} - \alpha^{*} a), \qquad \alpha = |\alpha| \mathrm{e}^{i\theta}, \tag{2.2}$$

where a (a^{+}) is the annihilation (creation) operator of the boson field.

The scalar product $\langle \beta, m|\alpha, n\rangle$ is given by [3]

$$\langle \beta, m|\alpha, n\rangle = \begin{cases} \langle\beta|\alpha\rangle\sqrt{\frac{n!}{m!}}(\alpha-\beta)^{m-n}L_n^{m-n}(|\alpha-\beta|^2), & m > n \\ \langle\beta|\alpha\rangle\sqrt{\frac{m!}{n!}}(\beta^{*}-\alpha^{*})^{n-m}L_m^{n-m}(|\alpha-\beta|^2), & n > m \end{cases} \tag{2.3}$$

where the scalar product of two coherent states has the well known value $\langle\beta|\alpha\rangle = \exp[-(1/2)(|\alpha|^2+|\beta|^2)+\alpha\beta^{*}]$, and $L_m^{\sigma}(x)$ is the associate Laguerre polynomial

$$L_m^\sigma(x) = \sum_{s=0}^{m} \begin{pmatrix} m+\sigma \\ m-s \end{pmatrix} \frac{(-x)^s}{s!}. \tag{2.4}$$

We consider the superposition of these states in the form [9]

$$|\Psi_m\rangle = A^{-\frac{1}{2}}\{|\alpha_0, m\rangle + K|-\alpha_0, m\rangle\}, \tag{2.5}$$

where A is the normalization constant given by

$$A = \{1 + |K|^2 + (K + K^*)\mathrm{e}^{-2|\alpha_0|^2} L_m(4|\alpha_0|^2)\}. \tag{2.6}$$

For $K = 0$ we have the DFS's, but for $K = 1$ or -1 the resulting states depend on m. If m is an even number and $K = 1$, we have superposition of even states and while odd states are obtained when $K = -1$. But when m is an odd number the result is reversed.

According to Cahill and Glauber [11] the P (Glauber–Sudarshan), W (Wigner) and Q (Husimi) functions may be expressed in an integral form

$$F(\alpha, s) = \frac{1}{\pi^2}\int C(\beta, s)\exp(\beta^*\alpha - \beta\alpha^*)\mathrm{d}^2\beta, \tag{2.7}$$

where $C(\beta, s)$ is the s-ordered generalized CF,

$$C(\beta, s) = \mathrm{Tr}[D(\beta)\rho]\exp\left(\frac{s}{2}|\beta|^2\right) \tag{2.8}$$

and s is a parameter which defines the relevant QDF's. According to the following values $s = 1, 0,$ and -1 the Glauber–Sudarshan P-function, the Wigner function, and the Q-function have been obtained, respectively.

3. Linear amplifier

There are three standard optical detection methods: heterodyne, homodyne, and direct detection which are realizations of the quantum measurements of photon number, single field quadrature, and both quadratures, respectively [15]. Correspondingly, there are three different optical amplifiers, the phase-insensitive linear amplifier, the phase-sensitive linear amplifier, and the photon number amplifier. The output of each of the amplifier preserves the measurement statistics of the input for the three different detection methods applied to the amplifier output [15].

An amplifier is a device that takes an input signal and produces an output signal by allowing the input signal to interact with the amplifier's internal degrees of freedom. A linear amplifier is one whose output signal is linearly related to the input signal. A phase-insensitive linear amplifier is one of the linear amplifiers whose phase shift of the input signal produces the same or opposite phase shift of the output signal, or the noise added by the amplifier is distributed randomly in phase [16]. The complete quantum mechanical description of linear amplifier can be found in ref. [16].

The used model is one which consists of a single-mode radiation field of frequency ω, which interacts with a large number of identical two-level atoms. The statistical properties of the field are governed by the following master equation for the density operator ρ of the field in the interaction picture

$$\frac{\partial\rho}{\partial t} = \eta N_2(2a^+\rho a - aa^+\rho - \rho aa^+) + \eta N_1(2a\rho a^+ - a^+a\rho - \rho a^+a), \quad (3.1)$$

where a and a^+ are the usual single-mode photon annihilation and creation operators, N_2 is the average population of the excited atoms and N_1 that of the unexcited atoms, and η denotes the coupling constant between the atoms and the field. The coupling constant is proportional to the square of the atomic transition matrix element with dimension (time) $^{-1}$ [13].

The equation of motion (3.1) can be converted to the Fokker–Planck equation with initial different field states [12]. The statistical properties of the linear amplifier oscillator are expressed exactly in terms of the QDF's. The linear amplifiers have been largely studied with coherent state input field [12–16]. Carusotto established the normally ordered CF of the output field for the amplifier and concluded that the output field is a superposition field of a thermal and an initial field [12]. Hillery and Yu [13] examined the linear amplifier of three kinds of higher-order squeezing and concluded that the fourth-order squeezing disappears at the output if the gain on the amplifier is greater than 2. The quantum statistical properties of the output field of linear light amplifier with squeezed state input field are reported in ref. [12a].

From eq. (3.1) Carusotto [12] was able to find the normally ordered CF, $C_N(\zeta, t)$, which is defined as

$$C_N(\zeta, t) = \mathrm{Tr}[\rho(t)\exp(\zeta a^+)\exp(-\zeta^* a)]. \quad (3.2)$$

It is found that

$$C_N(\zeta, t) = C_1(\zeta, t)C_2(\zeta, t), \quad (3.3)$$

where

$$C_1(\zeta, t) = \mathrm{Tr}[\rho(0)\exp(G^*\zeta a^+)\exp(-G\zeta^* a)] \quad (3.4)$$

and

$$C_2(\zeta, t) = \exp\left[-\frac{N_2}{N_2 - N_1}(|G|^2 - 1)|\zeta|^2\right] \quad (3.5)$$

with

$$G(t) = \exp[\eta(N_2 - N_1)t - i\omega t], \quad (3.6)$$

where $\rho(0)$ is the initial density operator, and ω the frequency of the field. It is clear that the system is an amplifier if $N_2 > N_1$. The quantity $|G|^2$ is the gain of the amplifier, in fact $|G|^2$ will be the gain, when $N_2 > N_1$, or the loss, when $N_1 > N_2$, factor.

For an input field on the linear amplifier, the superposition state $|\Psi_m\rangle$ which is assumed in the form of eq. (2.5) is chosen.

The density operator for an input state of the single-mode field given by the superposition of a pair of DFS's, takes the form

$$\rho(0) = |\Psi_m >< \Psi_m|. \tag{3.7}$$

By using the operators identities, we can write the s-ordered CF in this form,

$$\begin{aligned} C(\zeta, s, t) = \frac{1}{A} \exp &\left[\left\{ \frac{s-1}{2} + \frac{1}{2}|G|^2 - M(t) \right\} |\zeta|^2 \right] \left\{ \left[\exp \left\{ -\frac{1}{2}|\zeta|^2 \right\} L_m[|\zeta|^2] \right] \right. \\ &\left[\exp[G^* \alpha_0^* \zeta - G\alpha_0 \zeta^*] + |K|^2 \exp[-G^* \alpha_0^* \zeta + G\alpha_0 \zeta^*] \right] \\ &+ K \exp \left\{ -\frac{1}{2}|\zeta - 2\alpha_0|^2 \right\} L_m[|\zeta - 2\alpha_0|^2] \\ &\left. + K^* \exp \left\{ -\frac{1}{2}|\zeta + 2\alpha_0|^2 \right\} L_m[|\zeta + 2\alpha_0|^2] \right\}, \end{aligned} \tag{3.8}$$

where

$$M(t) = \frac{N_2}{N_2 - N_1}(|G|^2 - 1). \tag{3.9}$$

Thus, the s-parameterized CF is obtained; and from it, any expectation value for the field operators can be calculated.

3.1 *Moments*

The moments of the photon operators for the output of linear amplifier with superposition of DFS's as an initial state will be calculated. The s-ordered average value of a and a^+ can be calculated in the following way

$$\begin{aligned} \langle [a^{+k} a^l]_s \rangle &= \mathrm{Tr}[\rho \{a^{+k} a^l\}_s] \\ &= \frac{\partial^k}{\partial \zeta^k} \frac{\partial^l}{\partial (-\zeta^*)^l} C(\zeta, s, t)|_{\zeta = \zeta^* = 0} \end{aligned} \tag{3.10}$$

or through an integration involving the function $F(\beta, s)$.

The average values of the annihilation and creation operators are derived by differentiating the CF eq. (3.8) with respect to ζ and $-\zeta^*$, respectively:

$$\begin{aligned} \langle a^+ \rangle = \frac{G^*}{A} &\Big\{ (K - K^*) \Big[2\alpha_0^* \exp[-2|\alpha_0|^2] L_{m-1}^1 (4|\alpha_0|^2) \\ &+ \alpha_0^* \exp[-2|\alpha_0|^2] L_m(4|\alpha_0|^2) \Big] + \alpha_0^*(1 - |K|^2) \Big\} = (\langle a \rangle)^*. \end{aligned} \tag{3.11}$$

Similarly,

$$\begin{aligned} \langle a^+ a^+ \rangle = \frac{G^{*2}}{A} &\Big\{ (K + K^*) \Big[4(\alpha_0^*)^2 \exp[-2|\alpha_0|^2] L_{m-2}^2 (4|\alpha_0|^2) \\ &+ 4(\alpha_0^*)^2 \exp[-2|\alpha_0|^2] L_{m-1}^1 (4|\alpha_0|^2) \\ &+ (\alpha_0^*)^2 \exp[-2|\alpha_0|^2] L_m(4|\alpha_0|^2) \Big] + (1 - |K|^2)[\alpha_0^{*2}] \Big\} = (\langle aa \rangle)^*. \end{aligned} \tag{3.12}$$

The average number of photons can be acquired analogously:

$$
\begin{aligned}
\langle [a^+a]_s \rangle = \frac{|G|^2}{A}\Big\{ &(K+K^*)\Big[-4|\alpha_0|^2 \exp[-2|\alpha_0|^2] L^2_{m-2}(4|\alpha_0|^2) \\
&+\{(1-4|\alpha_0|^2\} \exp[-2|\alpha_0|^2] L^1_{m-1}(4|\alpha_0|^2) \\
&+\{|\alpha_0|^2 + A_1\} \exp[-2|\alpha_0|^2] L_m(4|\alpha_0|^2) \Big] \\
&+(1-|K|^2)[|\alpha_0|^2 + m - A_1] \Big\},
\end{aligned}
\tag{3.13a}
$$

where

$$
A_1 = \frac{-1}{|G|^2}\left\{ \frac{1-s}{2} + M(t) \right\}, \tag{3.13b}
$$

and $\langle a^+a^+aa\rangle$ can be analogously calculated.

The Glauber second-order coherence function is defined by

$$
g^{(2)} = \frac{\langle a^{+2}a^2\rangle}{\langle a^+a\rangle^2}. \tag{3.14}
$$

It has been classified that the light with $g^{(2)} < 1$ is a sub-Poissonian light, the light with $1 < g^{(2)} < 2$ is a super-Poissonian light, and the light with $g^{(2)} > 2$ is called super thermal light [18]. It is well known that the coherency is unity for the coherent light (Poissonian light). Substitution of $\langle a^+a^+aa\rangle$ and (3.13) into (3.14) yields the coherence function for the output of linear amplifier driven by superposition of DFS's.

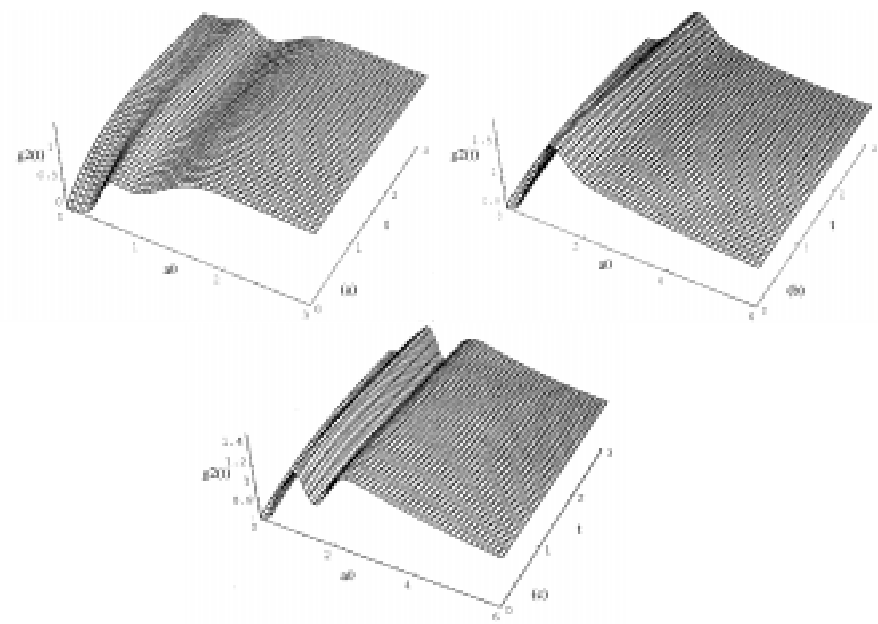

Figure 1. The coherence function $g^{(2)}(t)$ plotted against the interaction time t, and α_0, for $K = 1$ and the gain factor $|G| = \exp(0.2t)$. The number of photons have the values: (**a**) $m = 1$; (**b**) $m = 2$; (**c**) $m = 3$.

The autocorrelation function $g^{(2)}(t)$ of eq. (3.14) against the interaction time t and the displacement parameter α_0 is plotted in figure 1. We assume the parameters as follows: the gain factor $|G| = \exp(0.2t)$ (i.e., $\omega = 1$; $N_2 = 1$ and $N_1 = 0$) and the number of photons are assumed as: (a) $m = 1$; (b) $m = 2$; (c) $m = 3$. The constant K has the value $K = 1$. From figure 1, it is noted that the sub-Poissonian light exists for $t = 0$ and small α_0. Even though the case of light initially starts with sub-Poissonian distribution, it turns to Poissonian as α_0 develops. The super-Poissonian statistics exist when t develops and small α_0 with $m = 1$. A small difference appears in the behaviour of $g^{(2)}(t)$ for different values of the photon number m.

3.2 *Squeezing*

We study the squeezing properties of the superposition of DFS's. Moreover, the average values of the quadrature operators are presented. It is known that the quadrature operators of the single mode field are given by

$$X_1 = \frac{1}{2}(a + a^+), \qquad X_2 = \frac{1}{2i}(a - a^+) \tag{3.15}$$

such that $[X_1, X_2] = i/2$ which satisfies the uncertainty relation $\langle(\Delta X_1)^2\rangle\langle(\Delta X_2)^2\rangle \geq 1/16$ with the variance $\langle(\Delta X_j)^2\rangle = \langle X_j^2\rangle - \langle X_j\rangle^2$. The field is said to be squeezed if $(\Delta X_j)^2 < 1/4$ for (j=1 or 2).

The average values of the quadrature field operators and their variances $\langle X_1\rangle$, $\langle X_2\rangle$, $\langle(\Delta X_1)^2\rangle$ and $\langle(\Delta X_2)^2\rangle$ are directly computed.

The squeezing is best parameterized by

$$q_j = \frac{\langle(\Delta X_j)^2\rangle - 0.25}{0.25}, \qquad j = 1, 2 \tag{3.16}$$

such that squeezing exits for $-1 < q_j < 0$. Squeezing in one quadrature is achieved at the expense of increased noise in the conjugate quadrature; therefore, if one of q_j's is less than zero, then the other should be greater than zero.

In figure 2, we plot q_2 against the interaction time t and the displacement parameter α_0. The parameters are assumed: (a) $m = 1$ and $K = 1$; (b) $m = 2$ and $K = i$.

It is apparent that the degree of squeezing decreases with increasing t. The maximum squeezing in the case of $t = 0$ in figure 2a and $0.09 < \alpha_0 < 0.75$ is found. However, the maximum squeezing in figure 2b in the case of $t = 0$ and $0.09 < \alpha_0 < 0.25$.

Numerical calculations show that as the α_0 increases the squeezing degree of q_j decreases. From this one can conclude that the output of linear amplifier with superposition of pair of SCS's as initial state exhibits different nonclassical effects which depend on the particular choice of the phase θ_0.

4. s-Parameterized quasiprobability function

The QDF's for a quantum state of a physical system are useful tools for investigating the dynamical and statistical properties of a quantum mechanical system [11]. They include the

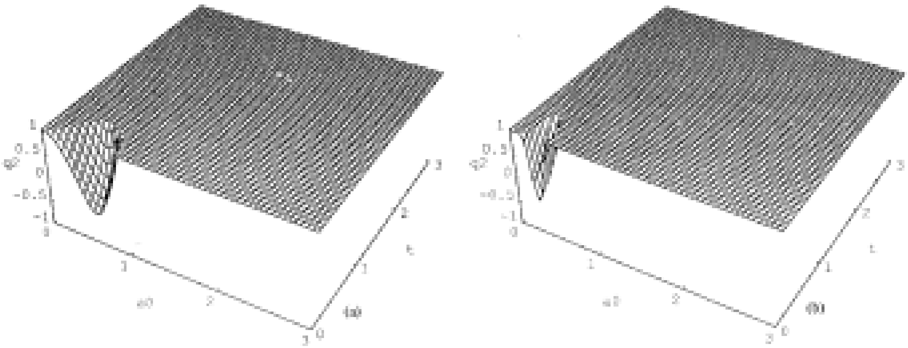

Figure 2. The plot of q_2 against the interaction time t and the displacement parameter α_0. The parameters are assumed: (**a**) $m = 1$ and $K = 1$; (**b**) $m = 2$ and $K = i$. The gain factor has the same value as in figure 1.

Glauber–Sudarshan P function, the Wigner W function and the Husimi Q function which are closely related to the operator ordering in the mathematical description of a physical system. The most widely used distributions are P function with the normal ordering, the W function with the symmetric or Weyl ordering, and the Q function with the antinormal ordering. These functions provide a way to characterize the non-classical nature of a quantum field. When the P function of a radiation field is not accepted as a classical probability density, the radiation field is said to be nonclassical, otherwise classical. They have now actually become accessible to measurements [17].

The s-ordered distribution functions are defined as a Fourier transformation of the s-ordered CF, and can be obtained by using (3.8) in (2.7). Upon performing the integration, the s-ordered distribution function for the output linear amplifier field may be written in the following form:

$$\begin{aligned} F(\beta,s,t) = {} & \frac{1}{\pi|G|^2 A(\nu_1)} \sum_{j=0}^{m} \binom{m}{j} \left(\frac{-1}{\nu_1}\right)^j \Bigg\{ \exp\left[\frac{-|\nu_2|^2}{\nu_1}\right] L_j\left[\frac{|\nu_2|^2}{\nu_1}\right] \\ & + |K|^2 \exp\left[\frac{-|\nu_3|^2}{\nu_1}\right] L_j\left[\frac{|\nu_3|^2}{\nu_1}\right] + K \exp\left[4\Delta|\alpha_0|^2 + 2\alpha_0^*\frac{\beta}{G} - 2\alpha_0\frac{\beta^*}{G^*}\right] \\ & \times \exp\left[\frac{\nu_4\nu_5}{\nu_1}\right] L_j\left[\frac{\nu_4\nu_5}{\nu_1}\right] + K^* \exp\left[4\Delta|\alpha_0|^2 - 2\alpha_0^*\frac{\beta}{G} + 2\alpha_0\frac{\beta^*}{G^*}\right] \\ & \times \exp\left[\frac{\nu_6\nu_7}{\nu_1}\right] L_j\left[\frac{\nu_6\nu_7}{\nu_1}\right] \Bigg\}, \end{aligned} \tag{4.1a}$$

where

$$\nu_1 = \frac{\{1 - s + 2M(t)\}}{2|G|^2}, \qquad \nu_2 = \left(\frac{\beta^*}{G^*} - \alpha_0^*\right), \tag{4.1b}$$

$$\nu_3 = \left(\frac{\beta^*}{G^*} + \alpha_0^*\right), \qquad \nu_4 = \left[2\Delta\alpha_0^* - \frac{\beta^*}{G^*}\right], \tag{4.1c}$$

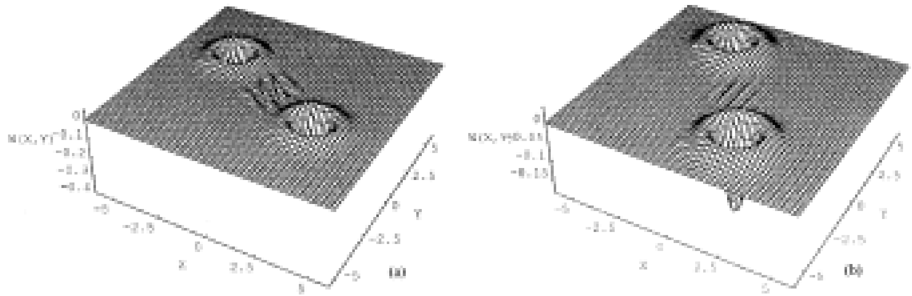

Figure 3. Three-dimensional time dependence of a Wigner distribution function for the output of the linear amplifier driven by the DFS's superposition with $\alpha_0 = 3$, $m = 1$ and $K = -1$. The amplifier parameters assume the same values in figure 1. The interaction time has the values: (**a**) $t = \pi/6$; (**b**) $t = \pi/3$. Here $X = \mathrm{Re}(\beta)$ and $Y = \mathrm{Im}(\beta)$.

$$\nu_5 = \left[2\Delta\alpha_0 + \frac{\beta}{G}\right], \qquad \nu_6 = \left[-2\Delta\alpha_0^* - \frac{\beta^*}{G^*}\right], \tag{4.1d}$$

$$\nu_7 = \left[-2\Delta\alpha_0 + \frac{\beta}{G}\right], \qquad \Delta = \frac{s-1}{2} + \frac{1}{2}|G|^2 - M(t). \tag{4.1e}$$

From this formula the exact analytical expressions for the s-parameterized QDF for the output linear amplifier with the superposition of the coherent states and Fock states can be found as special cases. It is noted that the P function, i.e., $s = 1$ exists for the output linear amplifier with the DFS's superposition states, when $t > \pi/2$.

In figure 3 plots the Wigner function, i.e., $s = 0$ with the parameters having the values: $m = 1$, $\alpha_0 = 3$, and $K = 1$. The amplifier parameter is $|G| = \exp(0.2t)$. The interaction time is assumed as: (a) $t = \pi/6$; (b) $t = \pi/3$. It is clear that (at $t = 0$) the Wigner function has two negative peaks observed which may be easily found at $x = \pm\alpha_0$ and the oscillatory regime between the two peaks, at $x = 0$. As time increases one can observe that the function rotates in the phase space and it spreads out with flattening of middle regime. The nonclassical nature of the linear amplifier driven by superposition of two DFS's is indicated by the negative values of the Wigner distribution function.

The Wigner functions of the output linear amplifier with the pair of DFS's superposition state as input for $m = 2$, $\alpha_0 = 3$ and $K = -1$ are shown in figure 4. The parameters have the same values as in figure 3. From the plots, two separated negative peaks and an oscillatory regime between them can be observed at $t = \pi/6$. The separation of the two peaks is seen to increase with α_0.

Generally, based on the numerical investigation, when $t = 0$ the behaviour of Wigner and the Q functions exhibit the standard distributions of the pair of DFS's superposition state as shown in ref. [12]. With increasing of time the maximum values of Wigner and Q functions decrease and rotate in a clockwise direction. The rotation in the phase space is due to the appearance of the frequency in the factor G. The spreading and shrinking of Wigner and Q functions over the β-plane is shown as time advances. The flattening of the two peaks are shown with time, which means an increase of diffusion as interaction time t

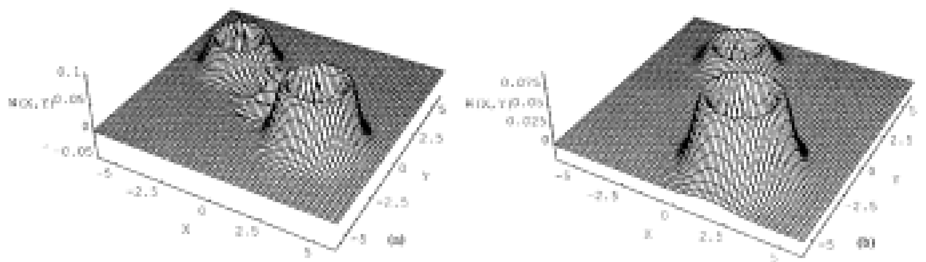

Figure 4a,b. The Wigner function (i.e., $s = -1$) for the output of the linear amplifier driven by the DFS's superposition with the same parameters in figure 3, but different excition number of photon $m = 2$.

progresses. As t becomes greater than 2π the various quasiprobability functions (i.e., P, W and Q functions) behave in nearly the same way, and they almost have the same shape.

5. Phase distribution

Recently, Barnett and Pegg defined the Hermitian phase operator in a finite dimensional state space [19]. They used the fact that, in this state space, one can define phase states rigorously. The phase operator is then defined as the projection operator on the particular phase state multiplied by the corresponding value of the phase. The main idea of the Pegg–Barnett (PB) formalism is based on the evaluation of all expectation values of physical variables in a finite dimensional Hilbert space. These give real numbers which depend parametrically on the dimension of the Hilbert space. Because a complete description of the harmonic oscillator involves an infinite number of states to be taken, a limit is taken only after the physical results (mean values of observables), are evaluated. This leads to a proper limit which corresponds to the results obtainable in ordinary quantum mechanics. It can be used to investigate the phase properties of quantum states of the single mode of the electromagnetic field [19].

Then finding the phase distribution of a quantum state is a nontrivial task. The reason for this is that Hermitian phase operators are rare [20]. However, one approach that is free of any such problems immediately offers itself. According to this approach, express one of the quasiprobability functions of this state that is in polar coordinates (radius and angle), and integrate it over the radius [21]. The resulting phase distribution is periodic in the phase angle. For various examples of states, it satisfies all properties required by a proper phase distribution.

The s-parameterized phase distribution $P(\theta, s, t)$ can be obtained by integrating the QDF, $F(\beta, s, t)$, over the radial variable $|\beta|$.

$$P(\theta, s, t) = \int_0^\infty F(\beta, s, t)|\beta|\mathrm{d}|\beta|. \tag{5.1}$$

By inserting eq. (4.1) in eq. (5.1) and using the integral form

$$\int_0^\infty x^{u-1}\exp(-vx^2 - \gamma x)\mathrm{d}x = (2v)^{-u/2}\Gamma(u)\exp\left[\frac{\gamma}{8v}\right] D_{-u}\left(\frac{\gamma}{\sqrt{2v}}\right), \tag{5.2}$$

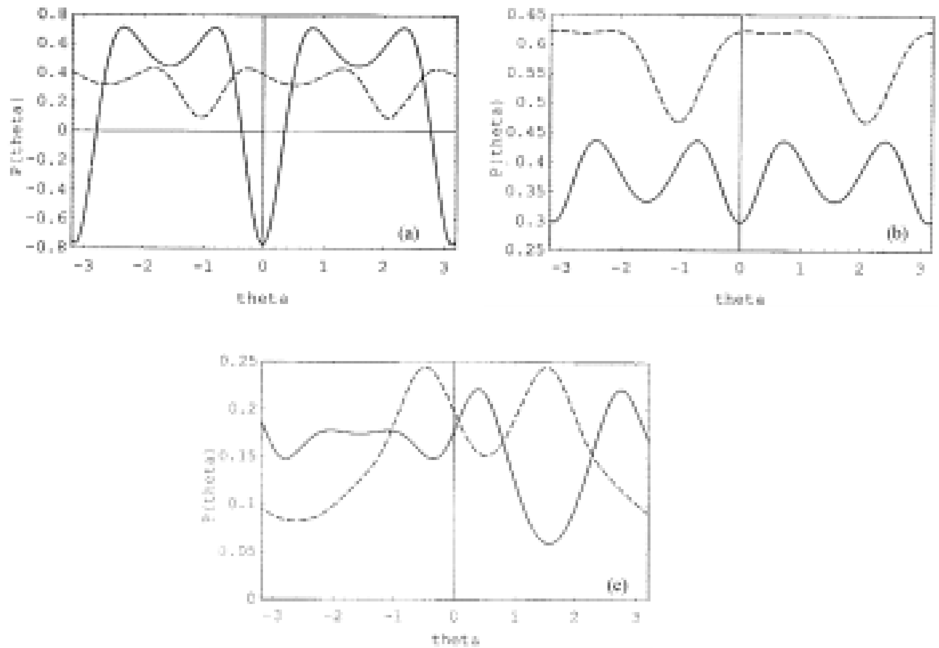

Figure 5. Pictures of phase distributions with $\alpha_0 = 1$, $m = 1$ and the amplifier parameter assume the same value in figure 1. The interaction t have the values: $t = 0$ solid curve and $t = \pi/3$ dashed curve. We have: **(a)** $P(\theta, 0, t)$, $K = 1$; **(b)** $P(\theta, -1, t)$, $K = -1$ and **(c)** $P(\theta, -1, t)$, $K = i$.

where $D_p(x)$ is the parabolic cylinder function [22]. Thereon, the phase distribution of eq. (5.1) may be calculated in a straightforward manner.

In figure 5a we plot the phase distribution $P(\theta, 0, t)$ obtained through the Wigner function ($s = 0$) for $m = 1$, $\alpha_0 = 1$, $|G| = \exp(0.2t)$, $\omega = 1$ and $K = 1$. The interaction time is assumed as: $t = 0$ for solid curve and $t = \pi/3$ for dashed curve.

Figure 5b and c show the plot of phase distribution $P(\theta, -1, t)$ produced through the Q function $(s = -1)$ with the same parameters in figure 5a, but with different values of K. The constant K is assumed: (b) $K = -1$; (c) $K = i$.

The phase distributions are exhibited in the double peak, with their splitting, appearing when $t = 0$. It is clear that the phase graph with two peaks is moving and the peaks become broader with increase of time. The phase information is lost as time develops. The high difference between the two peaks for $K = i$ in figure 5c.

The phase distributions $P(\theta, 0, t)$ have negative values for small t as it is noted in ref. [21] for the superposition of coherent states. But the $P(\theta, -1, t)$ does not have the negative values as may be expected.

It is seen that the phase distribution moves and broadens with the increase of time. A shift of the peak toward $-\pi$ is observed. The motion of the phase distribution is related to the rotation in the β-plane of the s-parameterized QDF in figures 3 and 4.

Numerical calculation shows that the Pegg–Barnett results lie between the $P(\theta, 0, t)$ and $P(\theta, -1, t)$ distributions. Nearly all three phase distributions give the same shape of two-peaks, with their splitting, but with differences: the sharpest peaks are those of $P(\theta, 0, t)$

and the broadest are those of $P(\theta,-1,t)$. The shape for all pictures is symmetric about $\theta=0,\pm\pi$. When we take $\alpha_0=|\alpha_0|e^{i\theta_0}$ and choose $\theta_0=\pi/2$ then the graph is symmetric about $\theta=\pm\pi/2$. The high values of α_0 make the phase distribution sharper.

6. Conclusions

The s-ordered CF and QDF for the output of linear amplifier with the pair of DFS's superposition state as input have been discussed. The formula for the s-ordered CF for the output of linear amplifier with the pair of DFS's superposition as an initial field have been obtained. Several moments have been calculated using the characteristic function as a function of the interaction time. The second-order correlation function $g^{(2)}(t)$ has been investigated numerically. The squeezing properties for these fields have been discussed. The three dimensional plots of the Wigner function for some parameters have been illustrated for the output of linear amplifier driven by the superposition of two DFS's showing nonclassical and interference effects. We have demonstrated the rotation of the QDF's in the phase space as a function of interaction time t. They have also been exhibited the asymmetrical diffusion for output fields. Our results generalize these in [12, 13, 23] for the linear amplifier. The physical interpretation of the output of linear amplifier with superposition of DFS's as an input tends to that for DFS's input as we have shown [12, 23] while time greater than π.

We have also obtained the phase distribution from the Q and Wigner QDF's. The behaviour of these distributions have been shown as functions of the interaction of time.

The present work was motivated by the desire to realize physically certain specific quantum states (superposition of DFS's) and use them as input for the linear-insensitive amplifier as one of its applications. It is hoped that the superposition of DFS's will find application in the quantum non-demolition measurements and quantum optics.

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