PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 54, No. 3
March 2000
pp. 441–445

On Hamiltonian formulation of cosmologies

K D KRORI*† and S DUTTA†
*Mathematical Physics Forum, Cotton College, Guwahati 781 001, India
†Address for correspondence: Shrimanta Shankar Academy, Guwahati 781 001, India

MS received 14 May 1999

Abstract. Novello *et al* [1,2] have shown that it is possible to find a pair of canonically conjugate variables (written in terms of gauge-invariant variables) so as to obtain a Hamiltonian that describes the dynamics of a cosmological system. This opens up the way to the usual technique of quantization. Elbaz *et al* [4] have applied this method to the Hamiltonian formulation of FRW cosmological equations. This note presents a generalization of this approach to a variety of cosmologies. A general Schrödinger wave equation has been derived and exact solutions have been worked out for the stiff matter era for some cosmological models. It is argued that these solutions appear to hint at their possible relevance in the early phase of cosmological evolution.

Keywords. Hamiltonian formulation; some cosmologies.

PACS No. 98.80. Hw

It has been shown by Novello *et al* [1,2] that it is possible to study perturbations in the FRW background in a gauge-invariant way in terms of pairs of observable variables, such as the electric and magnetic components of the conformal Weyl tensor, the shear and the vorticity. The authors have also pointed out the equivalence of their approach to the gauge-invariant variables introduced by Bardeen [3] to investigate cosmological perturbations.

It has also been shown [1,2] that it is possible to bring in a new pair of canonically conjugate variables (written in terms of the gauge-invariant variables) so as to obtain a Hamiltonian that describes the dynamics of the system under study. This paves the way to the usual technique of quantization.

Elbaz *et al* [4] have applied this method to the Hamiltonian formulation of FRW cosmological equations. The purpose of this note is to present a generalization of this approach to a variety of cosmologies.

We start with the following three fundamental cosmological equations [5]:

(a) *The Raychaudhuri equation*

$$\dot{\theta} + \frac{1}{3}\theta^2 + 2\sigma^2 + \frac{1}{2}(3p_0 + \rho) = 0, \tag{1}$$

where θ, σ, p_0 and ρ are expansion, shear, pressure and energy density respectively.

(b) *The generalized Friedmann equation*

$$\frac{1}{3}\theta^2 = \rho + \sigma^2 - \frac{1}{2}P, \tag{2}$$

where P is the scalar curvature of the homogeneous hypersurfaces which is always negative except in Bianchi IX.

(c) *The conservation equation*

$$\dot{\rho} + \theta(p_0 + \rho) = 0. \tag{3}$$

In view of the arbitrariness in the above three equations, we adopt the following relations:

$$p_0 = (\gamma - 1)\rho,$$
$$P = 2\mu\sigma^2, \tag{4}$$

where we call μ curvature parameter such that we have the Bianchi I and FRW models for $\mu \to 0$ and $\mu \to -\infty$ respectively. γ has the values $1 \leq \gamma \leq 2$.

Using (4) and (5) in (1)–(3) respectively we obtain

$$\dot{\theta} + A\theta^2 + B\rho = 0, \tag{5}$$

with

$$A = \frac{3-\mu}{3(1-\mu)} \quad \text{and} \quad B = \frac{1}{2}\left[\frac{(1-\mu)(3\gamma-2)-4}{(1-\mu)}\right],$$

$$\frac{1}{3}\theta^2 = \rho + (1-\mu)\sigma^2, \tag{7}$$

$$\dot{\rho} + \gamma\theta\rho = 0. \tag{8}$$

Now, we write the Hamiltonian, H, of the system in terms of a pair of canonically conjugate variables p and q as follows:

$$H = \frac{1}{2}p^2 + Cq^m, \tag{9}$$

where C and m are constants. p and q are defined [4] by

$$p = a\theta^\alpha\rho^\beta, \tag{10}$$
$$q = b\rho^\nu \tag{11}$$

and the Hamilton equations are

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{12}$$

Now, from (6)–(12) we obtain

$$a = bA, \quad \nu = -\frac{A}{\gamma}, \quad \beta = -\frac{A}{\gamma}, \quad \alpha = 1$$
$$m = \frac{2A-\gamma}{A}, \quad C = \frac{AB}{mb^{m-2}}, \tag{13}$$

so that we obtain from (10) and (11)

$$p = bA\theta\rho^{-A/\gamma}, \tag{14}$$
$$q = b\rho^{-A/\gamma}, \tag{15}$$

giving

$$\frac{p}{q} = A\theta. \tag{16}$$

Finally, the Hamiltonian takes the form

$$H = \frac{1}{2}p^2 + \frac{A^2Bb^{\gamma/A}}{2A-\gamma}q^{(2A-\gamma)/A}. \tag{17}$$

Now, following the usual quantization procedure (with $p \rightarrow i\partial_q$), we have the general Schrödinger equation

$$-\left(\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{2A^2Bb^{\gamma/A}}{\gamma-2A}q^{(2A-\gamma)/A}\right)\Psi = \xi\Psi. \tag{18}$$

Our next task is to solve (18) exactly for an early era, e.g., the stiff matter era ($\gamma = 2$), for a variety of cosmological models.

(a) *Model* 1 *(FRW)*. In this case, $\mu \rightarrow -\infty$ and $A = 1/3$ and $B = 2$ so that (18) is, with $a_0 = \frac{1}{3}b^6$ and $b_0 = \xi$,

$$q^4\Psi''(q) + [a_0 + b_0q^4]\Psi(q) = 0. \tag{19}$$

Now, with the substitution

$$\Psi(q) = x^{1/4}y(x), \quad x = q^2, \tag{20}$$

eq. (19) reduces to

$$x^3y'' + x^2y' + \left[\frac{a_0}{4} + \left(-\frac{1}{16}\right)x + \left(\frac{b_0}{4}\right)x^2\right]y = 0. \tag{21}$$

Next, with the substitution

$$y(x) = u(z), \quad mx = \mathrm{e}^{2iz}, \quad \frac{a_0m^2}{4} = \left(\frac{b_0}{4}\right), \tag{22}$$

we obtain the equation

$$u''(z) + (M + N\cos 2z)u(z) = 0, \tag{23}$$

where

$$M = \frac{1}{4}, \quad N = -2\sqrt{a_0 b_0}. \tag{24}$$

Equation (23) is the well-known Mathieu equation which has standard solutions [8].

(b) *Model* 2 *(Bianchi I)*. In this case, $\mu \to 0$ and $A = 1$, $(2A^2B/(\gamma - 2A)) = 3$ so that (18) is

$$\Psi''(q) + (\xi + 3b^2)\Psi(q) = 0, \tag{25}$$

with the simple solution

$$\Psi(q) = \Psi_0 \mathrm{e}^{i(\xi+3b^2)^{1/2}q}, \tag{26}$$

where Ψ_0 is a constant.

(c) *Model* 3. In this case, we take a set of values

$$A = 1/2, \quad \mu = -3, \quad B = 3/2 \tag{27}$$

so that (18) is now

$$q^2\Psi'' + \left(\frac{3}{4}b^4 + \xi q^2\right)\Psi = 0. \tag{28}$$

With $\frac{3}{4}b^4 = a(1-a)$, where a is $< (1/2)$,

$$q^2\Psi'' + [a(1-a) + \xi q^2]\Psi = 0. \tag{29}$$

Putting $\Psi = q^a y(z)$, $z = q^{1-2a}$, we have from (29)

$$y'' + Kz^r y = 0, \tag{30}$$

where $r = 4a/(1-2a)$ and $K = \xi/(1-2a)^2$.

Equation (30) has the solution

$$y = \sqrt{z} J_{1/p}\left[\frac{2\sqrt{k}}{p} z^{p/2}\right], \tag{31}$$

where $p = 2/(1-2a)$ and $J_{1/p}$ is a Bessel function of order $1/p$. Hence, the final solution is

$$\Psi = \sqrt{q} J_{1/p}\left[\sqrt{\xi} q\right]. \tag{32}$$

(d) *Model* 4. In this case we take a set of values

$$A = \frac{2}{3}, \quad \mu = -1, \quad B = 1 \tag{33}$$

so that (18) takes the form

$$q\Psi''(q) + \left[\frac{4b^3}{3} + \xi q\right]\Psi(q) = 0. \tag{34}$$

We put

$$\Psi(q) = \mathrm{e}^{i\xi^{1/2}x} y(x), \quad x = -2i\xi^{1/2} q \tag{35}$$

so that (34) takes the form

$$xy'' - xy' + \left(\frac{i4b^3}{6\xi^{1/2}}\right) y = 0. \tag{36}$$

This is a confluent hypergeometric equation. The solution is of the form

$$y = xF\left(1 + \frac{4b^3}{6i\xi^{1/2}},\ 2;\ x\right). \tag{37}$$

It may be noted that one may find a different model with another set of values of A, μ and B.

In fine, we would like to argue that the exact solutions derived here for some cosmological models for the stiff matter era hint, with their physical character, at their possible relevance in the early phase of cosmological evolution. The subject is obviously interesting and merits further investigation in more general cosmological contexts.

Acknowledgements

The authors express their profound gratitude to the Government of Assam for all facilities provided at Cotton College, Guwahati to carry out the investigation reported in this note.

References

[1] M Novello, J M Salim, M C Motta da Silva, S E Jorás and R Klippert, *Phys. Rev.* **D51**, 450 (1995)
[2] M Novello, J M Salim, M C Motta da Silva, S E Jorás and R Klippert, *Phys. Rev.* **D52**, 730 (1995)
[3] J Bardeen, *Phys. Rev.* **D22**, 1882 (1980)
[4] E Elbaz, M Novello, J M Salim, M C Motta da Silva and R Klippert, *Gen. Relativ. Gravit.* **29**, 481 (1997)
[5] J Ibañez, R J van den Hoogen and A A Coley, *Phys. Rev.* **D51**, 928 (1995)