PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 54, No. 3
March 2000
pp. 413–422

Random matrix model for disordered conductors

ZAFAR AHMED and SUDHIR R JAIN*
Nuclear Physics Division; *Theoretical Physics Division, Bhabha Atomic Research Centre, Mumbai 400 085, India

MS received 8 February 1999; revised 4 January 2000

Abstract. We present a random matrix ensemble where real, positive semi-definite matrix elements, x, are log-normal distributed, $\exp[-\log^2(x)]$. We show that the level density varies with energy, E, as $2/(1+E)$ for large E, in the unitary family, consistent with the expectation for disordered conductors. The two-level correlation function is studied for the unitary family and found to be largely of the universal form despite the fact that the level density has a non-compact support. The results are based on the method of orthogonal polynomials (the Stieltjes–Wigert polynomials here). An interesting random walk problem associated with the joint probability distribution of the ensuing ensemble is discussed and its connection with level dynamics is brought out. It is further proved that Dyson's Coulomb gas analogy breaks down whenever the confining potential is given by a transcendental function for which there exist orthogonal polynomials.

Keywords. Disordered conductors; random matrix theory; Dyson's Coulomb gas model.

PACS Nos 72.80.Ng; 02.50.Ey; 05.45.+b

1. Introduction

Matrix models are being successfully employed in a variety of domains of physics including studies on heavy nuclei [1], mesoscopic disordered conductors [2,3], two-dimensional quantum gravity [4], and chaotic quantum systems [5]. Universal conductance fluctuations in metals [6] and spectral fluctuations in chaotic quantum systems [7] find an excellent description in terms of random matrices when eigenvalues are in the scaling limit.

In this paper, our study is inspired by the problem of quantum electronic transport in disordered conductors. In terms of multiplicative transfer matrices, a random matrix formulation shows that [8] there can be true metallic systems only in dimensions greater than two, consistent with the one-parameter scaling theory of localization [9].

In the interpretation of transport properties of mesoscopic systems, the multichannel Landauer formula has been very useful. This formula expresses the dimensionless conductance g in terms of quantum-mechanical transmission amplitudes through a mesoscopic device. Landauer proposed that the conductance of a one-dimensional conductor between two phase-randomizing reservoirs is given by

$$g = \frac{e^2}{h}\frac{T}{R}, \tag{1}$$

where T and R are the transmission and reflection coefficients of the conductor treated as a single scattering centre, and where only one spin direction is included [10].

In the context of disorder-induced Anderson transition from a metal to an insulator, it becomes important to incorporate universal conductance fluctuations into the description. Conductance can be related to N non-degenerate eigenvalues (for N channels) $\lambda_n \geq 0$ of the matrix $\Lambda = 1/4[T^{\dagger}T + (T^{\dagger}T)^{-1} - 2]$:

$$g = 2\sum_{i=1}^{N} \frac{1}{1+\lambda_i}, \tag{2}$$

in units of e^2/h. One defines the random matrix model with N eigenvalues $0 \leq \lambda_n \leq \infty,\ n = 1, 2, ..., N$, whose distribution according to the maximum entropy hypothesis can be written as

$$P\{\lambda_1, \lambda_2, ..., \lambda_N\} \propto \prod_{m,n}^{N} |\lambda_m - \lambda_n|^{\alpha} \prod_{k=1}^{N} \exp[-V(\lambda_k)], \tag{3}$$

where α is the symmetry parameter. For simplicity, we restrict ourselves to the case of $\alpha = 2$ only.

The distribution (3) is inspired by an excellent agreement between two-point eigenvalue correlation function evaluated by random matrix model and by using the tight-binding Anderson model (ref. [12]). Since the conductance in one dimension is known to have a log-normal distribution [11], it is clear that the confining potential for large eigenvalues which describes the insulating regime must also behave as a square of its logarithm. It is this situation on which we shall concentrate in this paper. The distribution that naturally arises in the study of metal–insulator transitions in mesoscopic disordered conductors [12,13] is the one where the matrix elements are log-normal distributed.

It was noted by Dyson [14] that (3) is related to the partition function of a Coulomb gas of fermionic particles in two dimensions. This also forms the cornerstone of the theory of level dynamics which has been used to arrive at the probability distributions for random matrices [15,16]. In §2, we use the continuum approximation and starting from an average level density, we find the nature of the confining potential that binds (although weakly) the eigenvalues. This equation is approximate and its appearance simultaneously presents the question: is the nature of the confining potential always consistent with the logarithmic potential energy of a two-dimensional Coulomb gas? We show that the answer is negative. In fact, if the confining potential is given by any transcendental function, the Coulomb gas analogy fails almost always. Let us briefly understand the importance of this result. In the Coulomb gas analogy, the eigenvalues are taken as fictitious particles with two-body interaction in two dimensions. The fictitious particles are confined together in a potential well. To our knowledge, we have never found in published literature a condition or restriction on the form of the confining potential such that its presence does not alter the Coulomb interaction. In §2, we present this result with a simple proof.

Having obtained the confining potential corresponding to the level density, $2/(1+E)$, we present the random matrix ensemble in §3. Since the conductance fluctuations are known to be universal, and the average level density is modelled in the random matrix framework, our work here completes the modelling. Since it is shown in §2 that the Coulomb gas analogy fails, it is important to study the spectral fluctuations. In §4, we study the two-level correlation function.

Although the Coulomb gas analogy breaks down for any transcendental confinement, due to the diffusion analogy it is still possible to use part of the Dyson analogy, this subtle and interesting result is presented elsewhere [17]. Finally, in §5, we present a discussion summarizing our conclusions.

2. Confining potential and breakdown of Coulomb gas analogy

If a physical system is described by an $N \times N$ matrix H with eigenvalues, E_i ($a \leq E_i \leq b, i = 1, 2, ..., N$), within maximum entropy ansatz, the joint probability distribution function (jpdf) for the eigenvalues for an ensemble of random matrices, consistent with symmetries and the constraint of average level density, is

$$P(E_1, E_2, ..., E_N) = C_{N\alpha} \prod_{m<n}^{N} |E_m - E_n|^{\alpha} \prod_{k=1}^{N} \exp[-\alpha V(E_k)], \tag{4}$$

where we are concentrating in this paper at $\alpha = 2$, the (special) unitary case.

We are physically motivated, as discussed in the Introduction in detail, by the systems with average level density approximated by $2/(1+E)$ if a large number of levels are considered. From this, we need to obtain the confining potential, $V(E)$. To do this, let us observe that (4) can also be interpreted as the probability density of the position of N charges in two dimensions in a thermodynamic equilibrium at a temperature, $1/(k\alpha)$, where k is the Boltzmann constant. Thus,

$$\begin{aligned} P(E_1, E_2, ..., E_N) &= \exp[-\alpha W(E_1, E_2, ..., E_N)], \\ W(E_1, E_2, ..., E_N) &= \sum_{j=1}^{N} V(E_j) - \sum_{1 \leq i < j \leq N} \log |E_i - E_j|. \end{aligned} \tag{5}$$

The derivative of the potential, $V'(E)$, and $\sigma(E)$ are related via the Hilbert transform,

$$\mathcal{P} \int_0^{\infty} \mathrm{d}E' \frac{\sigma(E')}{E - E'} = \frac{\partial V(E)}{\partial E}. \tag{6}$$

The left hand side is a principal value integral. For $\sigma(E^{'}) = 2/(1+E')$, we note the Hilbert transform [18]

$$\mathcal{P} \int_0^{\infty} \frac{2}{1+E'} \frac{\mathrm{d}E'}{E - E'} = \frac{2 \log E}{1+E}. \tag{7}$$

For large E, using (6), we can write

$$\frac{\partial V(E)}{\partial E} \approx \frac{2 \log E}{E}, \tag{8}$$

which gives $V(E) = \log^2 E$.

We have now obtained the form of the confining potential, from where we can build a random matrix ensemble. Let us note that hitherto, (6) has been used for the cases of polynomial confining potentials (e.g., linear [22], quadratic [1]). For the time being, let us mention that if the form of density, $\sigma = 2/(1+E)$ is substituted in (6), numerically we see that the confining potential is similar to $\log^2(E)$ (continuous line in figure 2), but not the same (dashed line). This is the first indication that the Coulomb gas analogy is not exact in this case. We shall see below that, rigorously, it actually fails. Due to the fact that (3) (with $V(E) = \log^2(E)$) satisfies the diffusion equation, we can nevertheless use (6) locally [17].

Theorem. *If the matrix elements of a random matrix are distributed by a transcendental function so that the confining potential of eigenvalues in the continuum approximation is also a transcendental function* $V(E)$, *then the Coulomb gas model breaks down if there exist orthogonal polynomials with a weighting function* $\exp[-\alpha V(E)]$.

Proof: Using *Reductio ad absurdum*, let us assume that the Coulomb gas model works. Let the points, $E_1, E_2, E_3, \ldots$ be such as to minimize the potential energy,

$$W = \sum_{j=1}^{N} V(E_j) - \sum_{1\le i\le j\le N} \log|E_i - E_j|. \tag{9}$$

Then,

$$0 = -\frac{\partial W}{\partial E}(E_j) = -\frac{\partial V}{\partial E}(E_j) + \sum_{i\neq j} \frac{1}{E_j - E_i}. \tag{10}$$

Let us assume that there exist orthogonal polynomials with $e^{-\alpha V(E)}$ as the weighting function. For the case where $V(E) = \log^2(E)$, we shall see in the next section that the orthogonal polynomials needed are the Stieltjes–Wigert (SW) polynomial, $p_N(E)$. Let E_j be the zeros of the SW polynomial. By the fundamental theorem of algebra,

$$p_N(E) = C(E-E_1)(E-E_2)(E-E_3).....(E-E_N), \tag{11}$$

C being a constant. Next, we can write [20]

$$\frac{p_N''(E_j)}{2p_N'(E_j)} = \sum_{i\neq j} \frac{1}{E_i - E_j}. \tag{12}$$

Substituting in (10), we get a differential equation for $p_N(E)$,

$$\frac{p_N''(E_j)}{2p_N'(E_j)} - \frac{2\log E_j}{E_j} = 0, \tag{13}$$

which is absurd for a ratio of two polynomials and can never be a transcendental function. This concludes the proof. Thus, for $V(E) = \log^2(E)$ also, the Coulomb gas analogy breaks down.

3. Random matrix ensemble: Level density

In accordance with the discussion above, we begin by introducing an ensemble of random matrices where the matrix elements are distributed log-normally. Thus, a matrix element,

H_{ij}, of a random matrix are distributed by $P(H_{ij}) = \mathcal{N} \exp(-\log^2 H_{ij})$. The jpdf of the N eigenvalues is given by

$$P(E_1, E_2, ..., E_N) = C_{N\alpha} \prod_{m<n}^{N} |E_m - E_n|^{\alpha} \prod_{k=1}^{N} \exp[-\alpha V(E_k)], \tag{14}$$

where $V(E_k) = \log^2 E_k$. The polynomials, $p_n(E)$ associated with a log-normal weighting function,

$$W(E) = \frac{k}{\sqrt{\pi}} \exp[-k^2 \log^2(E)], \tag{15}$$

satisfy the relation

$$\int_0^{\infty} \mathrm{d}E W(E) p_n(E) p_m(E) = \delta_{mn}. \tag{16}$$

The Stieltjes–Wigert polynomials [19], $p_n(E)$, are given by

$$\begin{aligned} p_n(E) = (-1)^n q^{(n/2)+(1/4)} \{(1-q)(1-q^2)...(1-q^n)\}^{-1/2} \\ \times \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} q^{\nu^2+(\nu/2)} (-E)^{\nu}, \end{aligned} \tag{17}$$

where $q = \exp[-(2k^2)^{-1}]$, and the Gauss polynomial

$$\begin{bmatrix} n \\ \nu \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})...(1-q^{n-\nu+1})}{(1-q)(1-q^2)...(1-q^{\nu})}, \quad 0 < \nu < n \tag{18}$$

is used. For $\nu = 0$ and n, this symbol yields unity. Also, for $n = 0$, the quantity in braces in (17) must be replaced by unity. Once these polynomials are known, the level correlations can be computed using standard formulae [1].

The normalization constant, which involves the Selberg-type integral, is

$$C_{N\alpha}^{-1} = (A_1 A_2 A_3, \ldots, A_N)^2, \tag{19}$$

where

$$A_n = q^{\{n+(1/2)\}^2} \{(1-q)(1-q^2)...(1-q^n\}^{-(1/2)}. \tag{20}$$

This constant has also been found earlier [21].

For $N \times N$ matrices in the unitary family, the level density is given by

$$\sigma_N(E) = \sum_{j=0}^{N-1} p_j^2(E). \tag{21}$$

We have plotted $\sigma_N(E)$ from (21) for $N = 100$ in figure 1, which fits very well with a ratio of two polynomials below, and also agrees well with $2/(1+E)$ asymptotically. That is

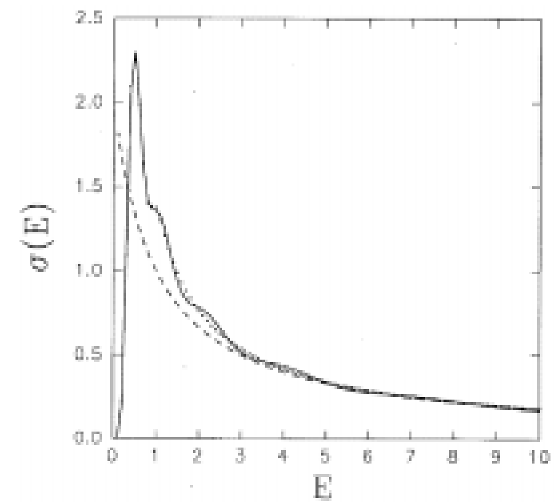

Figure 1. Density of levels for $N = 100$ as a function of energy is plotted. The solid line is the random matrix result, (21), dotted line corresponds to the ratio of polynomials in (22), and the dashed line is $2/(1+E)$. Beyond the energies, $E = 5$, all the results agree with each other.

$$\sigma_N(E) = \frac{\mathrm{a} + \mathrm{b}E + \mathrm{c}E^2 + \mathrm{d}E^3 + \mathrm{e}E^4}{1 + \mathrm{f}E + \mathrm{g}E^2 + \mathrm{h}E^3 + \mathrm{i}E^4 + \mathrm{j}E^5} \tag{22}$$

$$\sim \frac{2}{1+E}, \tag{23}$$

where the coefficients are written in [23].

Zeros of orthogonal polynomials are very interesting and useful. In passing, we would like to point out that in a similar context, in the past [13], while finding zeros of the Stieltjes–Wigert polynomials, the authors of [13] actually cited the zeros (q^{-l}, $l = 1, 2, ..., n$) of the following polynomial

$$\sum_{\nu=0}^{\infty} \begin{bmatrix} n \\ \nu \end{bmatrix} q^{\frac{\nu(\nu+1)}{2}} (-E)^{\nu} = (1 - qE)(1 - q^2E)...(1 - q^nE). \tag{24}$$

If now, the more exact form of $\sigma_N(E)$ (given as a ratio of polynomials, (22) or (21) itself) is used and (6) is employed, the confining potential, discussed in §2, turns out to agree very well with $\log^2 E$ (see dotted curve in figure 2). Also, when for density, $2/(1+E)$ is used in (6), we get a confining potential (dashed curve of figure 2) which is similar to $\log^2 E$, but not the same.

Some studies on density of levels in a generalized unitary ensemble have been done in the past [24]. The level densities found there are quite different, one which has a tail is of a Gaussian form. However, the universal properties are different as in their work, there is a parameter, μ, which is allowed to scale with the size of the matrix.

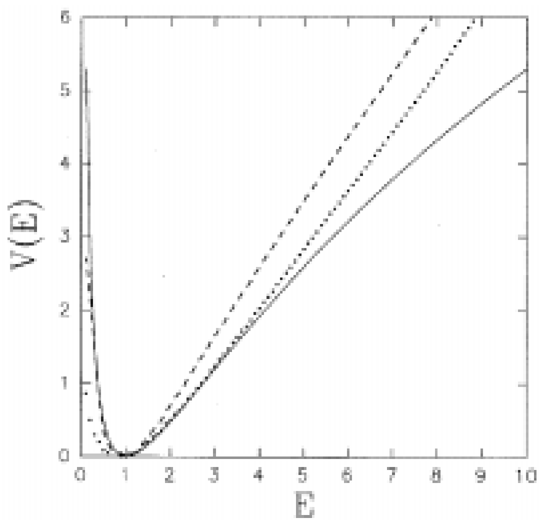

Figure 2. The confining potential is related to the level density through the Dyson equation. Here, these potentials are given for the three densities shown in figure 1 (solid line: $\log^2 E$, dotted line: using $\sigma(E) = 2/(1+E)$, dashed line: using the ratio of polynomials (22)).

4. Two-level correlation function

The Dyson two-level correlation function, $T_2(E-S/2, E+S/2)$ is given by

$$T_2\left(E-\frac{S}{2}, E+\frac{S}{2}\right) = \left[\frac{K\left(E-\frac{S}{2}, E+\frac{S}{2}\right)}{\sigma(E)}\right]^2 \tag{25}$$

$$= \frac{\sin^2(\frac{\pi S}{\overline{S}})}{(\frac{\pi S}{\overline{S}})^2}, \tag{26}$$

which is a celebrated expression of universality as the second equality is independent of E in the scaling limit. In (26), $\overline{S}$ is the mean level spacing. In figure 3, we see this universality for $E = 2$ (this value is purely incidental). Referring to figure 1, we see that at $E = 1$ or $E = 2$, the actual level density (21) does not agree well with the continuum approximation, $2/(1+E)$. We also notice the deviation from the universal result in the wings in figure 3a, denoted by open circles. The agreement between solid line and open circles is a semi-analytic proof of universality. In figure 3b, the universality is clearly seen for large energies (here, $E = 10$). It may be noted that in the context of the Hofstadter model, for generalized Laguerre ensembles non-universal correlations have been found and studied [25].

Interestingly enough, the above universal result agrees also with the expression in terms of the orthogonal polynomials, viz.,

$$K(E, E') = \sum_{j=0}^{N-1} p_j(E) p_j(E'). \tag{27}$$

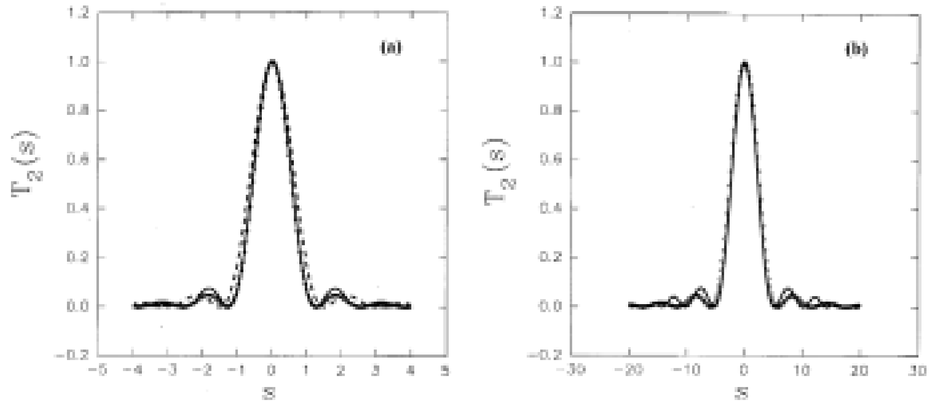

Figure 3. (**a**) The two-level correlation function is given by (25). The solid line corresponds to the method of orthogonal polynomials where $K(E, E')$ is given by (27). The universal correlation (26) for mean level spacings corresponding to the level densities from orthogonal polynomials, and, $2/(1+E)$ are given by open circles and dashes respectively. The energy E is 2. (**b**) Same as figure 3a for $E = 10$.

A simple, yet important, point should be borne in mind: since $E \in [0, \infty]$, $E \pm (S/2) \geq 0$, thus $S \in [-2E, 2E]$, which explains the abscissae of the plots.

It is expected from the works of Brèzin and Zee [26], and Ambjorn and collaborators [27], that eigenvalue correlations are independent of the weighting function, when suitably scaled. Since the weighting function or the confining potential is connected with the average level density, the correlations are independent of the level density. We shall see that the Coulomb gas model works in general if the jpdf satisfies an N-dimensional diffusion equation. This is clearly not always the case. For the model here, this holds and so does the universality. It has been recently shown [28] that all local correlations are universal and independent of the confining potential if the support of the spectrum is finite in the limit, $N \to \infty$. Going beyond, in the present example, the support is infinite and the local correlations are universal.

5. Conclusions

We arrive at the following conclusions:

1. We have shown in this paper that a log-normal weighting function leads to a density of eigenvalues that is similar to the one that occurs for disordered conductors [29]. With heuristic arguments, using the ideas pertaining to Dyson Coulomb gas analogy, we obtained the confining potential for the levels (eigenvalues of a random matrix).
2. Rigorously, we have shown that the Coulomb gas picture breaks down whenever the confining potential is given by a transcendental function from which orthogonal polynomials can be constructed.
3. For the matrix with elements distributed log-normally, we have used the Stieltjes–Wigert polynomials to show the form of level density. The level density has a tail given by $2/(1+E)$. This is of great interest in the theory of disordered conductors.

4. The universality of level correlations is seen for large E (see figures 3(a) and (b)) to agree well with the usual random matrix result for the unitary ensemble. For these values of E, it can also be seen that the level density is connected to the confining potential through the Dyson equation. So, the universality of level correlations is connected to the Coulomb gas analogy. In general, as we have shown above, for this ensemble, this analogy breaks down.
5. It has been shown recently that all local correlations are universal [28] and independent of the confining potential if the support of the spectrum is finite in the limit $N \to \infty$. We have seen here that the two-level correlations are universal (figure 3b) when energy is taken large so that the level density is well-approximated by $2/(1+E)$. Clearly, the support of the spectrum is not finite.

References

[1] M L Mehta, *Random matrices* (Academic Press, New York, 1991)
[2] G Montambaux, in *Quantum fluctuations* Les Houches, Session LXIII edited by E Giacobino, S Reynaud and J Zinn-Justin (Elsevier, Netherlands, 1995)
[3] Y Imry, in *Directions in condensed matter physics* (World Scientific, Singapore, 1992)
[4] *Two-dimensional quantum gravity and random surfaces* edited by D J Gross, T Piran and S Weinberg (World Scientific, Singapore, 1992)
[5] S W McDonald, Ph.D. Thesis (UMI-8413506, Ann Arbor, MI, 1983)
O Bohigas, M-J Giannoni and C Schmit, *Phys. Rev. Lett.* **52**, 1 (1984)
S R Jain and S V Lawande, *Proc. Indian Natl. Sci. Acad.* **65**, 275 (1995)
[6] S Washburn and R A Webb, *Adv. Phys.* **35**, 375 (1986)
[7] T H Seligman and H Nishioka, *Quantum chaos and statistical nuclear physics*, Lecture notes in Physics (Springer, Heidelberg, 1986) vol. 263
[8] K A Muttalib, *Phys. Rev. Lett.* **65**, 745 (1990)
[9] P A Lee and T V Ramakrishnan, *Rev. Mod. Phys.* **57**, 287 (1985)
[10] R Landauer, *IBM J. Res. Develop.* **1**, 233 (1957); *Philos. Mag.* **21**, 863 (1970)
[11] P W Anderson, D J Thouless, E Abrahams, and D S Fisher, *Phys. Rev.* **B22**, 3519 (1981)
V I Mel'nikov, *Sov. Phys. Solid State* **23**, 444 (1981)
[12] K A Muttalib, Y Chen, M E H Ismail and V N Nicopoulos, *Phys. Rev. Lett.* **71**, 471 (1993)
[13] Y Chen, M E H Ismail and K A Muttalib, *J. Phys.* **C5**, 177 (1993)
[14] F J Dyson, *J. Math. Phys.* **13**, 90 (1972)
[15] P Pechukas, *Phys. Rev. Lett.* **51**, 943 (1983)
[16] T Yukawa, *Phys. Rev. Lett.* **54**, 1883 (1985)
[17] S R Jain and Z Ahmed, *Brownian motion model of random matrices revisited* (preprint) 1999
[18] A Erdelyi, W Magnus, F Obberhetinger and F G Tricomi (eds), *Tables of integral transforms* Bateman manuscript project (McGraw-Hill Inc., New York, 1954) vol. 2, p. 244
[19] G Szegö, *Orthogonal polynomials*, AMS Colloquium Publications (AMS, Providence, Rhode Island, 1939) vol. XXIII
[20] Appendix 6 in ref. [1]
[21] P Forrester, *J. Stat. Phys.* **76**, 331 (1994)
[22] B V Bronk, *J. Math. Phys.* **6**, 228 (1965)
[23] The coefficients are $\mathrm{a} = 0.063, \mathrm{b} = -1.690, \mathrm{c} = 12.279, \mathrm{d} = -22.123, \mathrm{e} = 12.003, \mathrm{f} = -5.912, \mathrm{g} = 13.020, \mathrm{h} = -8.111, \mathrm{i} = -5.677, \mathrm{j} = 6.068$
[24] M Kreynin and B Shapiro, *Phys. Rev. Lett.* **74**, 4122 (1995)

[25] T Nagao and K Slevin, *J. Math. Phys.* **34**, 2075; 2317 (1993)
[26] E Brezin and A Zee, *Nucl. Phys.* **B402**, 613 (1993)
[27] J Ambjorn, J Jurkiewicz and Yu M Makeenko, *Phys. Lett.* **B251**, 517 (1990)
[28] G Hackenbroich and H A Weidenmüller, *Phys. Rev. Lett.* **74**, 4118 (1995)
[29] C Blecken, Y Chen and K A Muttalib, *J. Phys.* **A27**, L563 (1994)