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Strong curvature naked singularities in generalized Vaidya space-times

K D PATIL*, R V SARAYKAR and S H GHATE
Department of Mathematics, Nagpur University Campus, Nagpur 440 010, India
*College of Engineering, Sewagram, Wardha, India

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Abstract. In this paper, following recent results on generalized Vaidya solutions by Wang, we prove that under certain conditions on generalized mass function, strong curvature naked singularities exist in radiation collapse in monopole-Vaidya space-times and also in charged-Vaidya space-times. We thus unify and generalize results of Dwivedi–Joshi and Lake–Zannias. The general case also covers de Sitter–Vaidya space-time recently treated by Wagh–Maharaj with a view to study existence of naked singularities.

Keywords. Radiation collapse; naked singularity; strong curvature.

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1. Introduction

Proof of cosmic censorship conjecture is one of the most important open problems in general relativity. Existence of naked singularities occurring in some examples of spherically symmetric gravitational collapse (see [1], chapters 6 and 7) is, by now, a well established phenomenon. Some of these singularities are strong curvature ones and hence while formulating a provable version of cosmic censorship conjecture, it is essential to avoid such singularities or to avoid those initial data which evolve into such singularities. One of the earlier examples of such naked singularities was studied by Papapetrou (see [1]) by considering the collapse of null radiations in Vaidya space-time with a linear mass function. This example was studied later by many workers in detail (see references in [1]). Recently Wang [2] introduced a more general family of Vaidya space-times which covers monopole solutions, de Sitter, and anti de Sitter solutions and charged Vaidya solutions as special cases. In this paper, we show that, as in the null radiation collapse in Vaidya space-time, strong curvature naked singularities arise in generalized Vaidya solutions also, namely in the monopole Vaidya case and in the charged Vaidya case. This is proved under physically reasonable conditions on mass function. We thus generalize earlier results of Dwivedi and Joshi [3] and unify them with the results of Lake and Zannias [4]. We discuss de Sitter–Vaidya case as it has been studied recently by Wagh and Maharaj [5].

Thus, in §2, we briefly describe generalized Vaidya space-times following Wang [2]. We also describe geodesic equations for those space-times that are needed in later sections. In §3, we study collapse of radiation shells and prove the existence of strong curvature naked singularities in monopole Vaidya solutions. In §4, we derive similar results for charged Vaidya solutions. We make some concluding remarks on general mass functions in non-self-similar generalized Vaidya space-times.

2. Generalized Vaidya space-times

Generalized Vaidya space-times are described by the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(u,r)}{r}\right)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{1}$$

where $m(u,r)$ is a mass function. The energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + P)(l_\mu n_\nu + l_\nu n_\mu) + P g_{\mu\nu} + \mu l_\mu l_\nu,$$

where l_μ, n_μ are two null vectors,

$$l_\mu = \delta^0_\mu, n_\mu = \frac{1}{2}\left(1 - \frac{2m(u,r)}{r}\right)\delta^0_\mu - \delta^1_\mu, l_\lambda l^\lambda = n_\lambda n^\lambda = 0, \quad l_\lambda n^\lambda = -1,$$

and

$$\mu = \frac{2\dot{m}(u,r)}{\kappa r^2}, \quad \rho = \frac{2m'(u,r)}{\kappa r^2}, \quad P = -\frac{m''(u,r)}{\kappa r}, \tag{2}$$

κ being gravitational constant, the dot and prime denote partial derivatives with respect to u and r respectively. The energy conditions are
(a) the weak and strong energy conditions

$$\mu \geq 0, \quad \rho \geq 0, \quad P \geq 0, \quad (\mu \neq 0),$$

b) the dominant energy condition:

$$\mu \geq 0, \quad \rho \geq P \geq 0, \quad (\mu \neq 0).$$

When $\rho = P = 0$, the space-times reduce to the Vaidya space-time with $m = m(u)$ and the energy condition (weak, strong, and dominant) reduces to $\mu > 0$. Therefore, the above space-times are regarded as generalized Vaidya space-times.

The situation being considered here is that of a radially injected radiation flow in an initially empty region of Minkowski space-time. The radiation is focussed into a central singularity at $r = 0$, $u = 0$, of growing mass $m(u,r)$. Thus $m(u,r)$ is an arbitrary non-negative increasing function of u and r. When the source of radiation is turned off after a finite time u_s, at some value of $r = r_s$, the space-time settles to the Schwarzschild space-time. Thus, $m(u,r)$= constant for $u > u_s$ and $r > r_s$, some fixed values of u and r.

Following the method of Dwivedi and Joshi [3], we now examine under what conditions on $m(u,r)$ such a collapse leads to a naked singularity and whether this singularity is a strong curvature one or not. For explanation of 'strong curvature singularity', we refer the reader to [1].

The existence of naked singularity or otherwise can be determined by examining the behaviour of radial null geodesics. If they terminate at the singularity in the past with a definite positive tangent vector, then the singularity is naked. If the tangent vector is not positive in the limit as one approaches the singularity, then the singularity is covered. From the above metric, geodesic equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}k}(K^u) - \left(\frac{m'}{r} - \frac{m}{r^2}\right)(K^u)^2 - r((K^\theta)^2 + \sin^2\theta(K^\phi)^2) = 0, \tag{3}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}(K^r) + &\left(\frac{m}{r} - \frac{m'}{r} + \frac{m}{r^2} + \frac{2mm'}{r^2} - \frac{2m^2}{r^3}\right)(K^u)^2\\ &-(1-2m/r)r((K^\theta)^2 + \sin^2\theta(K^\phi)^2)\\ &+2\left(\frac{m'}{r} - \frac{m}{r^2}\right)K^uK^r = 0,\end{aligned} \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}k}(K^\theta) + \frac{2}{r}K^rK^\theta - \sin\theta\cos\theta(K^\phi)^2 = 0, \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}k}(r^2\sin^2\theta K^\phi) = 0, \tag{6}$$

where $(K^u, K^r, K^\theta, K^\phi)$ i.e. ($\mathrm{d}u/\mathrm{d}k$, $\mathrm{d}r/\mathrm{d}k$, $\mathrm{d}\theta/\mathrm{d}k$, $\mathrm{d}\phi/\mathrm{d}k$) are components of a tangent vector and k is the affine parameter. Solving (5) and (6), we get

$$K^\phi = \frac{c}{r^2\sin^2\theta} \quad \text{and} \quad K^\theta = \frac{1}{r^2}\left(c_1 - \frac{c^2}{\sin^2\theta}\right)^{1/2}, \tag{7}$$

where c and c_1 are constants of integration. Putting $c_1 = l^2$ and $c = l\cos\beta$, we get

$$K^\phi = \frac{l\cos\beta}{r^2\sin^2\theta} \quad \text{and} \quad K^\theta = \frac{l}{r^2\sin\theta}(\sin^2\theta - \cos^2\beta)^{1/2}, \tag{8}$$

where l is the impact parameter and β is the isotropy parameter. We also have

$$(K^\theta)^2 + \sin^2\theta(K^\phi)^2 = \frac{l^2}{r^4} \tag{9}$$

If K^a is a null vector, we have $K_aK^a = 0$. Using this and equations (3), (4), and (9), we get

$$\left(1 - \frac{2m}{r}\right)(K^u)^2 - 2K^uK^r = \frac{l^2}{r^2}. \tag{10}$$

Let

$$K^u = \mathrm{d}u/\mathrm{d}k = R(u,r)/r. \tag{11}$$

Then (10) implies

$$K^{r} = \left(1 - \frac{2m}{r}\right)\frac{R}{2r} - \frac{l^{2}}{2rR}, \tag{12}$$

where R satisfies the differential equation

$$\frac{\mathrm{d}R}{\mathrm{d}k} - \frac{R^{2}}{2r^{2}}\left(1 - \frac{4m}{r}\right) - \frac{l^{2}}{2r^{2}} - \frac{R^{2}}{r^{2}}m' = 0. \tag{13}$$

To examine the behaviour of a non-space-like curve near the singularity, we need to find the explicit expression for R. However, in general, complete integration of the above geodesic equations is a difficult task. Hence to analyse the nature of the singularity, we restrict ourselves to the study of behaviour of radial null geodesics ($l = 0$) and some special nature of the mass function $m(u,r)$.

3. Naked singularities in monopole-Vaidya space-times

Monopole solutions in Vaidya background are given by

$$2m(u,r) = \lambda u + ar$$

where λ and a are arbitrary constants.

With this mass function, the metric (1) admits a homothetic Killing field given by

$$\xi^{a} = r\frac{\partial}{\partial r} + u\frac{\partial}{\partial u}.$$

In this case, $\xi^{a}K_{a} = A$ (constant) along null geodesics. After some computations we get

$$\begin{aligned} R &= \frac{A\{1 \pm (1 + (2 + \lambda X^{2} + aX - X)L^{2}X)^{1/2}\}}{(2 + \lambda X^{2} + aX - X)} \\ &= \frac{A(1 \pm S)}{2 + \lambda X^{2} + aX - X}, \end{aligned} \tag{14}$$

where

$$X = \frac{u}{r},\ L = \frac{l}{A} \quad \text{and} \quad S = (1 + (2 + \lambda X^{2} + aX - X)L^{2}X)^{1/2} \tag{15}$$

Consider the limit of function X as we approach the singularity ($r = 0, u = 0$) along a radial null geodesic:

$$X_{0} = \lim_{\substack{r \to 0 \\ u \to 0}} X = \lim_{\substack{r \to 0 \\ u \to 0}} \frac{u}{r} = \lim_{\substack{r \to 0 \\ u \to 0}} \frac{\mathrm{d}u/\mathrm{d}k}{\mathrm{d}r/\mathrm{d}k} = \lim_{\substack{r \to 0 \\ u \to 0}} \frac{K^{u}}{K^{r}}.$$

Thus X_{0} gives the tangent at the singularity. If X_{0} is positive, then the singularity will be naked and if X_{0} is negative or zero, singularity will be covered.

Using (11), (12), (14), and (15), we get, after some algebra,

$$X_0 = \frac{2}{1 - \lambda X_0 - a} \quad \text{or equivalently}$$

$$X_0 = \frac{(1-a) \pm \sqrt{(1-a)^2 - 8\lambda}}{2\lambda}.$$

Thus, since $\lambda > 0$ (for weak energy condition to be satisfied) we have X_0 positive if $0 < a < 1$ and $0 < \lambda < ((1-a)^2)/8$. So the singularity will be naked if

$$0 < a < 1 \quad \text{and} \quad 0 < \lambda < ((1-a)^2)/8. \tag{16}$$

In general in the expression for $m(u,r)$, the term ar can be replaced by an arbitrary function $f(r)$ of r, and a in the above expression will denote $\lim_{r\to 0} f(r)/r$.

To see whether this singularity is a strong curvature one or not, we follow the argument given in [1].

A sufficient condition for a strong curvature naked singularity is that

$$\lim_{k\to 0} k^2 R_{ab} K^a K^b \neq 0,$$

where k is an affine parameter. If we compute this for the case of $m(u,r)$ under consideration, we get (since S has a definite limit ($S \to 1$)) at the singularity,

$$\lim_{k\to 0} k^2 R_{ab} K^a K^b = \frac{4\lambda}{(1-a)^2} > 0.$$

This shows that the naked singularity arising in this space-time is a strong curvature one.

4. Naked singularities in charged Vaidya space-time

This is another special class of generalized Vaidya space-times, and is given by

$$2m(u,r) = 2\lambda u - \frac{\lambda u^2}{r}, \tag{17}$$

where $\sqrt{\lambda}u$ denotes the electric charge. General electric charge is denoted by $q(u)$ and in this case $2m(u,r) = 2\lambda u - q^2(u)/r$. The most general expression for charged Vaidya solution is given with $m(u,r) = f(u) - q^2(u)/(2r)$ and the condition $\mu \geq 0$ gives the main restriction on the choice of the functions f and q. In particular, if $\mathrm{d}f/\mathrm{d}q > 0$, there always exists a critical radius r_c such that when $r < r_c$, we have $\mu < 0$, where $r_c = (q\mathrm{d}q/\mathrm{d}u)/(\mathrm{d}f/\mathrm{d}u)$, and thus the energy condition seems to be violated. However, from the equation of motion for the massless charged particles that consist of the charged null fluid, it follows that the hypersurface $r = r_c$ is a vanishing point, and due to the repulsive Lorentz force, the 4-momenta of the particles vanish exactly on $r = r_c$. The Lorentz force will then push the particles to move outwards. Therefore, in realistic situations the particles cannot get into the region $r < r_c$ and thus the energy conditions are satisfied.

The metric with mass function given by (17) admits a homothetic Killing field as in §3 and hence can be dealt with easily. The solution R of D.E. (13) is given by (with notations of §3)

$$R = \frac{A\{1 \pm (1 + (2 - X + 2\lambda X^2 - \lambda X^3)L^2 X)^{1/2}\}}{(2 - X + 2\lambda X^2 - \lambda X^3)}, \tag{18}$$

hence X_0 is the tangent to the radial null geodesic at the singularity, given by

$$X_0 = \frac{2}{1 - 2\lambda X_0 + \lambda X_0^2} \quad \text{which implies}$$

$$(\lambda X_0^2 + 1)(X_0 - 2) = 0. \tag{19}$$

Thus $X_0 = 2$ gives a naked singularity. The other case $\lambda X_0^2 + 1 = 0$ gives

$$X_0 = \pm(-1/\lambda)^{1/2}$$

and is imaginary for $\lambda > 0$. If λ is negative, then X_0 becomes real and the charge becomes imaginary, an unphysical situation. Thus λ should be positive. In any case $X_0 = 2$ gives a naked singularity.

To find the strength of this singularity, we again compute $R_{ab}K^aK^b$ and find that

$$R_{ab}K^aK^b = \frac{2\dot{m}}{r^2}(K^u)^2 = \frac{2\lambda(1 - X)R^2}{r^4}.$$

Taking

$$\lim_{k\to 0} k^2 R_{ab}K^aK^b,$$

we get

$$\lim_{k\to 0} k^2 R_{ab}K^aK^b = \frac{8\lambda(1 - q_+)}{(1 - \lambda q_+^2)^2},$$

where q_+ is the positive value of X_0. Since $X_0 = 2$, this becomes

$$\frac{-8\lambda}{(1 - 4\lambda)^2} \neq 0.$$

Thus naked singularity appearing here is also a strong curvature one.

With the general expression for charge $q(u)$ as a function of u, existence of naked singularity or otherwise, will depend on the nature of function $q(u)$, and on how X_0 appears in the limit of $q^2(u)/r^2$ as $u \to 0$ and $r \to 0$. Also in this case, since the metric, in general, will not admit a homothetic Killing field, it is not possible to evaluate R explicitly as a solution of D.E. (13). However, to examine whether such a space-time contains a naked singularity or not, we can proceed by writing geodesic equation for a radial null geodesic directly from the metric (1):

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{2r}{r - 2m(u, r)}.$$

The existence of a naked singularity then depends on the behaviour of the functions $\dot{m}$ and m' in the limit as $u \to 0$ and $r \to 0$.

This is true of generalized Vaidya solutions also, and not only for charged Vaidya solution. We can proceed along the lines of Joshi and Dwivedi [6] or Lake [7] and draw conclusions about the existence of naked singularity and its strength.

Wagh and Maharaj [5] recently examined deSitter–Vaidya space-time and proved the existence of a naked singularity provided $0 < \lambda < 1/8$. Here, $2m(u,r) = \lambda u - \Lambda r^3/3$, and Λ is cosmological constant. This is a special case of $f(r)$, namely $f(r) = \Lambda r^3/3$ mentioned at the end of §3 where $a = \lim_{r\to 0} f(r)/r = 0$. General considerations discussed above are to be applied to this case since de Sitter–Vaidya space-time does not admit homothetic Killing vector field. This is discussed in detail in [5]. The conclusion of this paper is that whether the space-time is asymptotically flat or not, it does not make any difference to the occurrence of a naked singularity. This is true at least in the case of collapsing radiation shells.

We hope that in future, taking into consideration all such examples showing the existence of naked singularities, a provable version of cosmic censorship conjecture is formulated.

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