PRAMANA — journal of physics

Vol. 58, No. 1
January 2002
pp. 113–119

Test particle trajectories near cosmic strings

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MS received 26 February 2001; revised 17 August 2001

Abstract. We present a detailed analysis of the motion of test particle in the gravitational field of cosmic strings in different situations using the Hamilton–Jacobi (H–J) formalism. We have discussed the trajectories near static cosmic string, cosmic string in Brans–Dicke theory and cosmic string in dilaton gravity.

Keywords. Cosmic strings; test particles; Brans–Dicke fields; dilaton gravity.

PACS Nos 04.20.Jb; 98.80.Cq; 04.50; 97.60.Lf

1. Introduction

Phase transitions of quantum fields in the early universe produced very thin tubes of false vacuum, known as cosmic strings [1]. These are topological defects which can form when either a local or a global symmetry is spontaneously broken in a phase transition. The first one is called gauge string and the latter is called global string. Gauge strings have their energy concentrated in a very thin tube, the radius of which is of the order of the symmetry-breaking scale whereas the global strings are such that their energy extends to regions far beyond the central core. Strings have an important astrophysical consequence, namely, the double quasar problem, galaxy formation can well be explained by strings [1]. Vilenkin [2] has shown that cosmic strings can act as a gravitational lens and hence astronomical observations may detect this object.

So many works were done on global strings [1–3]. In 1996, Banerjee *et al* [4] have found solutions for a static $U(1)$ global string. This involves only a complex scalar field. Recently Sen *et al* [5] and Gregory *et al* [6] have described solutions for global string in Brans–Dicke theory and global string in dilaton gravity respectively.

In this work, we have studied the motion of a test particle in the gravitational field of global strings using the approach of Chakraborty [7] and examined whether Brans–Dicke coupling parameter and dilaton coupling parameter could be restricted to obtain bound orbit.

This study will be of relevance to the structure formation because it gives us some idea about the behaviour of the particles (created at the early universe) in the gravitational field of the cosmic strings.

The paper is organized as follows: In §2, we have considered the motion of a test particle near static global string. In §3, we have shown that Brans–Dicke field has effect on the trajectory of the test particle near global string. In §4, we have investigated the behaviour of a test particle in the gravitational field of a cosmic string in dilaton gravity. The paper ends with a short discussion in §5.

2. Static global string

The line element describing static global string is given by [5]

$$\mathrm{d}s^2 = A(-\mathrm{d}t^2 + \mathrm{d}z^2) + B(\mathrm{d}U^2 + \mathrm{d}\theta^2), \tag{1}$$

where

$$A = \left(\frac{U}{U_0}\right); \quad B = \gamma^2 \left(\frac{U_0}{U}\right)^{1/2} \exp(U_0^2 - U^2) \tag{2}$$

with U_0 and γ as constants.

We consider a relativistic particle with mass m moving in the gravitational field of a static global string (1). The H–J equation has the expression [7]

$$-\frac{1}{A}\left(\frac{\partial s}{\partial t}\right)^2 + \frac{1}{A}\left(\frac{\partial s}{\partial z}\right)^2 + \frac{1}{B}\left(\frac{\partial s}{\partial U}\right)^2 + \frac{1}{B}\left(\frac{\partial s}{\partial \theta}\right)^2 + m^2 = 0. \tag{3}$$

In order to solve this first order partial differential equation, let us use separation of variables for the H–J function S as follows:

$$s(U, z, \theta, t) = -Et + S_1(U) + J\cdot\theta + M\cdot z. \tag{4}$$

Here the constants J and E are the angular momentum and energy of the particle respectively and M is the momentum along z-direction.

If we substitute the ansatz (4) in (3), then the expression for unknown function $S_1(U)$ is

$$S_1(U) = \varepsilon \int \left(\frac{BE^2}{A} - \frac{BM^2}{A} - J^2 - m^2 B\right)^{1/2} \mathrm{d}U. \tag{5}$$

Now the trajectory can be obtained following H–J method as [7]

$$\frac{\partial s}{\partial E} = \text{ constant}; \quad \frac{\partial s}{\partial M} = \text{ constant}; \quad \frac{\partial s}{\partial J} = \text{ constant}. \tag{6}$$

Thus we get

$$A = \varepsilon \int \frac{EB}{A} \left(\frac{BE^2}{A} - \frac{BM^2}{A} - J^2 - m^2 B\right)^{-1/2} \mathrm{d}U, \tag{7}$$

$$\theta = \varepsilon \int J \left(\frac{BE^2}{A} - \frac{BM^2}{A} - J^2 - m^2 B\right)^{-1/2} \mathrm{d}U, \tag{8}$$

$$z = \varepsilon \int \frac{MB}{A} \cdot \left(\frac{BE^2}{A} - \frac{BM^2}{A} - J^2 - m^2 B \right)^{-1/2} \mathrm{d}U. \tag{9}$$

(We have taken constants in (6) to be zero without any loss of generality and $\varepsilon = \pm 1$).

From (7), we obtain the radial velocity of the particle as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{A}{EB} \left(\frac{BE^2}{A} - \frac{BM^2}{A} - J^2 - m^2 B \right)^{1/2}. \tag{10}$$

So the turning points of the trajectory are given by $(\mathrm{d}U/\mathrm{d}t) = 0$. As a consequence the potential curves are

$$\frac{E}{m} = \left(\frac{M^2}{m^2} + \frac{U^{3/2} \mathrm{e}^{U^2}}{\gamma^2 U_0^{3/2} \mathrm{e}^{U_0^2}} + \frac{U_0}{U} \right)^{1/2}. \tag{11}$$

We see that the extremals of the potential curve are the solutions of the equation

$$f(U) \equiv \mathrm{e}^{U^2} \cdot U^{5/2} \left(\frac{3}{2} + 2U^2 \right) - U_0^{5/2} \mathrm{e}^{U_0^2} \cdot \gamma^2 = 0. \tag{12}$$

Since $f(U)$ is not simple, we rewrite equation $f(U) = 0$ as

$$g(U) \equiv \mathrm{e}^{U^2} = \frac{U_0^{5/2} \mathrm{e}^{U_0^2} \gamma^2}{U^{5/2}(3/2 + 2U^2)} \equiv h(U). \tag{13}$$

Now we draw the graphs of $y = g(U)$ and $y = h(U)$ with respect to rectangular axes. Then the U-coordinates of the point of intersection of the graphs give the crude approximation of the real root of the equation $f(U) = 0$. (The constant $U_0^{5/2} \cdot \mathrm{e}^{U_0^2} \cdot \gamma^2$ is taken to be unity) (see figure 1). Thus real extremal exists. Hence the trajectory of the test particle is bounded, i.e. the particle can be trapped by static global string.

3. Global string in Brans–Dicke theory

The metric ansatz describing the gravitational field of a global string in Brans–Dicke theory is [5]

$$\mathrm{d}s^2 = A(-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2) + B\mathrm{d}\theta^2, \tag{14}$$

where $A = B_1^{-1/m} r^{-1/m}$ and $B = B_1^b r^b$ with

$$m = \frac{4\omega + 7}{2}, \quad b = \frac{2A_1 + 1}{m}, \quad A_1 = 2\omega + 2, \quad B_1 = \frac{1}{2} \nu \frac{(4\omega + 7)}{\sqrt{2\omega + 3}},$$

$$\nu = \text{constant}. \tag{15}$$

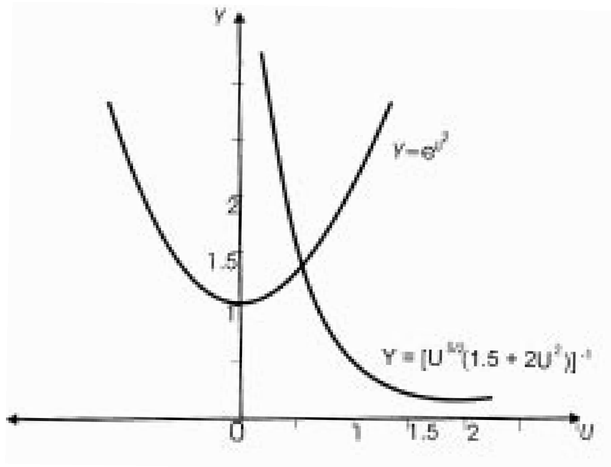

Figure 1.

To study the motion of a test particle of mass m_0, we shall begin with the H–J equation,

$$-\frac{1}{A}\left(\frac{\partial s}{\partial A}\right)^2+\frac{1}{A}\left(\frac{\partial s}{\partial z}\right)^2+\frac{1}{A}\left(\frac{\partial s}{\partial r}\right)^2+\frac{1}{B}\left(\frac{\partial s}{\partial \theta}\right)^2+m_0^2=0. \tag{16}$$

The solution of H–J equation can be expressed in the following form

$$s(t,z,r,\theta)=-E\cdot t+S_1(r)+J\cdot\theta+M\cdot Z, \tag{17}$$

where constants E,J,M have the similar meaning as above. Putting (17) in (16) we get

$$S_1(r)=\varepsilon\int\left(E^2-M^2-\frac{A}{B}J^2-m_0^2A\right)^{1/2}\mathrm{d}r. \tag{18}$$

The trajectory of the particle is characterized by

$$t=\varepsilon\int E\left(E^2-M^2-\frac{A}{B}J^2-m_0^2A\right)^{-1/2}\mathrm{d}r, \tag{19}$$

$$z=\varepsilon\int M\left(E^2-M^2-\frac{A}{B}J^2-m_0^2A\right)^{-1/2}\mathrm{d}r, \tag{20}$$

$$\theta=\varepsilon\int \frac{AJ}{B}\left(E^2-M^2-\frac{A}{B}J^2-m_0^2A\right)^{-1/2}\mathrm{d}r. \tag{21}$$

Thus from (19) the radial velocity of the particle is

$$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{1}{E}\left(E^2-M^2-\frac{A}{B}J^2-m_0^2A\right)^{1/2}. \tag{22}$$

The turning points of the trajectory (where $(\mathrm{d}r/\mathrm{d}t) = 0$) are

$$\frac{E}{m_0} = \left[\frac{M^2}{m_0^2} + \frac{J^2}{m_0^2} B_1^{-(2A_1+2)/m} \cdot r^{-(2A_1+2)/m} + B_1^{-1/m}\, r^{-1/m}\right]^{1/2}. \tag{23}$$

The extremal is given by

$$r = (-1)^{(4\omega+7)/(8\omega+10)} \cdot \frac{(4\omega+6)^{(4\omega+7)/(8\omega+10)}(4\omega+3)^{1/2}}{\frac{1}{\sqrt{2}}\nu(4\omega+7)}. \tag{24}$$

We see that if $((4\omega+7)/(8\omega+10))$ is an even integer, a real extremal exists for the potential curve. So, the trajectory of the test particle is trapped by string.

In this case we note that $((4\omega+7)/(8\omega+10)) = 2K$ (where K is a positive integer), i.e.

$$\omega = \frac{7-20K}{16K-4}. \tag{25}$$

Hence for bound orbit ω must be negative. We also note that when $\omega \to \infty$, r becomes imaginary. We know that cosmic string in general relativity has gravitational effect on the matter around it but here the particle cannot be trapped by cosmic string in Brans–Dicke theory when $\omega \to \infty$.

So Brans–Dicke cosmic string metric does not reduce to cosmic string in general relativity when $\omega \to \infty$. This result is in agreement with the result of Sen *et al* [5].

4. Cosmic string in dilaton gravity

We now consider cosmic string in dilaton gravity with metric ansatz of the form [6]

$$\mathrm{d}s^2 = A(\mathrm{d}t^2 - \mathrm{d}r^2 - \mathrm{d}z^2 - B\mathrm{d}\theta^2), \tag{26}$$

where

$$A = r^{(a+1)\varepsilon\frac{\bar{\mu}}{2} + (a+1)^2\varepsilon^2\frac{\bar{\mu}^2}{4}} \quad \text{and} \quad B = (1-\varepsilon\bar{\mu})^2 r^{2-2(a+1)^2\varepsilon\frac{\bar{\mu}^2}{4}} \tag{27}$$

with $\bar{\mu} = \frac{\mu}{4\pi\varepsilon}$, $\varepsilon = \frac{\eta^2}{2}$, $\mu =$ energy per unit length of the string, $\eta \sim 10^{15}$ GeV and a is a parameter which determines the strength of the coupling between the string Lagrangian and the dilaton field.

In this case the H–J equation has the expression

$$\frac{1}{A}\left(\frac{\partial s}{\partial t}\right)^2 - \frac{1}{A}\left(\frac{\partial s}{\partial r}\right)^2 - \frac{1}{AB}\left(\frac{\partial s}{\partial \theta}\right)^2 - \frac{1}{A}\left(\frac{\partial s}{\partial z}\right)^2 + m^2 = 0. \tag{28}$$

The solution can be written as

$$s = -E\cdot t + S_1(r) + J\cdot\theta + M\cdot Z. \tag{29}$$

Here $S_1(r)$ has the expression

$$S_1(r) = \varepsilon' \int \left(E^2 - M^2 - \frac{J^2}{B} + m^2 A \right)^{1/2} \mathrm{d}r. \tag{30}$$

The trajectory of the particle is characterized by

$$A = \varepsilon' \int E \left(E^2 - M^2 - \frac{J^2}{B} + m^2 A \right)^{-1/2} \mathrm{d}r, \tag{31}$$

$$\theta = \varepsilon' \int \frac{J}{B} \left(E^2 - M^2 - \frac{J^2}{B} + m^2 A \right)^{-1/2} \mathrm{d}r \tag{32}$$

$$= \varepsilon' \int M \left(E^2 - M^2 - \frac{J^2}{B} + m^2 A \right)^{-1/2} \mathrm{d}r. \tag{33}$$

Here $\varepsilon' = \pm 1$. Hence the expression for radial velocity is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2} \left(E^2 - M^2 - \frac{J^2}{B} + m^2 A \right)^{1/2}. \tag{34}$$

The turning points are given by

$$\frac{E}{m} = \left[\frac{M^2}{m^2} + \frac{J^2 r^{2(a+1)^2 \varepsilon^2 \bar{\mu}^2 - 2}}{m^2 (1 - \varepsilon \bar{\mu})^2} - r^{(a+1)\varepsilon(\bar{\mu}/2) + (a+1)^2 \varepsilon^2 (\bar{\mu}^2/4)} \right]^{1/2} \tag{35}$$

which determines the potential curves.

The extremal is given by

$$r = \left[\frac{(a+1)\varepsilon \frac{\bar{\mu}}{2} + (a+1)^2 \varepsilon^2 \bar{\mu}^2 \cdot m^2 (1 - \varepsilon \bar{\mu})^2}{J^2 (2(a+1)^2 \varepsilon^2 (\bar{\mu}^2/4) - 2)} \right]^{\frac{1}{(a+1)^2 \varepsilon^2 (\bar{\mu}^2/4) - (a+1)\varepsilon(\bar{\mu}/2) - 2}}. \tag{36}$$

Hence bound orbits are possible provided

$$(a+1)\varepsilon \bar{\mu} \neq 2. \tag{37}$$

5. Summary

In this paper, we have studied the motion of test particles in the gravitational field of cosmic strings in three different situations.

We have seen that the particles can be trapped by the gravitational field of the static global string without any restriction on the parameters. But in the last two cases for bound orbits, we have to impose some restrictions. The trajectory of the test particle can be trapped by the global string in Brans–Dicke theory due to the restriction on the coupling parameter ω. Here one has to take ω to be negative. But as $\omega \to \infty$, r becomes imaginary and hence particle cannot be trapped by the string. Also the metric (14) does not reduce to the corresponding general relativity case for $\omega \to \infty$ as we know the global string in general relativity has gravitational effect on particles [3]. We have seen that the cosmic string in dilaton gravity may have gravitational field with certain restrictions (see eq. (37)). Chakraborty *et al* [8] studied the trajectory of the test particles near the cosmic string in different situations and showed that the particles can be trapped in some cases. So for future work it will be interesting to study the gravitational effect on the test particles near different cosmic strings.

References

[1] A Vilenkin, *Phys. Rep.* **121**, 263 (1985)
A Vilenkin and E P S Shellard, *Cosmic string and other topological defects* (Cambridge University Press, 1994)
[2] A Vilenkin, *Phys. Rev.* **D23**, 852 (1981)
[3] W A Hiscok, *Phys. Rev.* **D31**, 3288 (1985)
D Harari and P Sikivie, *Phys. Rev.* **D37**, 3448 (1988)
A K Ray Choudhuri, *Phys. Rev.* **D41**, 3041 (1990)
R Gregory, *Phys. Lett.* **B215**, 663 (1988)
A D Cohen and D B Kaplan, *Phys. Lett.* **B215**, 65 (1980)
R Gregory, *Phys. Rev.* **D39**, 2108 (1989)
E Shaver and K Lake, *Phys. Rev.* **D40**, 3287 (1989)
[4] A Banerjee, N Banerjee and A A Sen, *Phys. Rev.* **D56**, 3706 (1997)
[5] A A Sen, N Banerjee and A Banerjee, *Phys. Rev.* **D53**, 5508 (1996)
[6] R Gregory and C Sauton, *Phys. Rev.* **D56**, 1194 (1997)
[7] S Chakraborty, *Gen. Relativ. Gravit.* **28**, 1115 (1996)
[8] S Chakraborty and L Biswas, *Class. Quant. Grav.* **13**, 2153 (1996)