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Quantum mechanics for two-timers

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Abstract. Extensions of standard quantum mechanics with joint probability distributions for position coordinates and momenta have been proposed in the literature. Time is assumed to be one-dimensional in these studies. In view of recent interest in two-dimensional time, the construction is extended to this situation and found to satisfy the necessary consistency conditions.

Keywords. Two-timer; consistency condition; joint probability distribution.

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1. Introduction

This is a meeting on FQT, whereas I am a QFT person, so I found it hard to decide what to talk about. It was suggested that I could talk on *black holes and quantum mechanics*, but I thought it would be inappropriate to talk about quantum gravity here. I will talk about something else, hoping that it will not be considered as quantum levity.

One ordinarily thinks of space as having three dimensions and time only one. There have been many theoretical constructions involving four or more spatial dimensions. The work of Kaluza and of Klein are quite well known. I believe everyone has heard of string theorists who talk about twenty-six dimensional or ten or eleven dimensional spacetime. In spite of the large number of spacetime dimensions, time is still kept one-dimensional here. But recently Bars and followers [1,2] have started talking about two-dimensional time. Their own motivation is theoretical and I shall not go into it here. But there is also some psychological motivation for exploring the possibility of two-dimensional time. It often happens that one feels time to be passing very slowly or very fast. If one does not look at the watch for a while and tries to guess how much time has elapsed, one finds one's estimate widely different from the indication on the watch. This can be understood if the two times are *different*, which can happen if time forms a plane with two dimensions or possibly a higher dimensional space.

Two-dimensional time can be of special interest to workers on foundations of quantum theory. Two-dimensional time allows the possibility of *holes*. Time evolution along a curve in the time-plane may avoid a region of singularity which may be associated with a measurement-like process. However, this is a bit hypothetical at the moment and my theme for today is different.

Standard quantum mechanics asserts that one cannot assign simultaneous values of a non-commuting pair of variables to any system. Thus it is not possible to have a quantum system in an eigenstate of position and momenta. However, it is possible to have a joint probability distribution for two non-commuting variables without violating the uncertainty principle. Joint probability distributions have been suggested by Wigner [3], de Broglie and Bohm [4], and recently by Roy and Singh [5]. Such distributions evolve in time, and this evolution, which is determined by the quantum Hamiltonian of the system, may also be understood in terms of a different *classical* Hamiltonian. Today's talk aims at the investigation of this Hamiltonian formalism in the situation when time is two-dimensional.

First we shall consider some simple facts about two-dimensional time.

2. Classical evolution

Two times require two Hamiltonians for evolution of a system. The Hamiltonian equations of motion get altered to

$$\begin{aligned} \frac{\partial q}{\partial t_a} &= \frac{\partial H_a}{\partial p}, \\ \frac{\partial p}{\partial t_a} &= -\frac{\partial H_a}{\partial q}. \end{aligned} \tag{1}$$

Here $a = 1, 2, ...$ for two or more times. The phase space coordinates can be multidimensional, of course, as usual, but this has not been explicitly indicated in the equations. For consistency of the above equations, one needs:

$$\frac{\partial H_1}{\partial t_2} - \frac{\partial H_2}{\partial t_1} = \{H_2, H_1\}. \tag{2}$$

This requirement arises because the mixed derivative of a q or a p with respect to t_1 and t_2 may be calculated in two different orders but they have to match. In case the Hamiltonians are time-independent, the requirement is simply that they have a vanishing Poisson bracket.

3. Quantum evolution

When one goes to quantum mechanics, time-dependent Schrödinger equations are needed. There are two of them because there are two times.

$$i\frac{\partial \psi}{\partial t_a} = H_a \psi. \tag{3}$$

For consistency one again needs a condition:

$$\frac{\partial H_1}{\partial t_2} - \frac{\partial H_2}{\partial t_1} = \frac{[H_2, H_1]}{i}. \tag{4}$$

This can be understood from the requirement of consistency of the two mixed derivatives of the wave-function with respect to the two times calculated in two different orders. It can also be obtained from the classical consistency condition by replacing the Poisson bracket by a commutator in the standard manner.

We shall now briefly review the introduction of joint probability distributions for variables normally regarded as incompatible, namely those with vanishing Poisson brackets at the classical level, and having noncommuting operators at the quantum level. This goes beyond quantum mechanics and is, at the moment, untested.

4. Joint probability distribution for x and p

A class of such distributions is described by the formula

$$\rho(x,p,t) = \rho_x(x,t)\delta(p - p_x(x,t)). \tag{5}$$

Here $\rho_x(x,t) = |\psi(x,t)|^2$ is the standard probability distribution function for x when the system is in a state given by the wave-function ψ. The non-standard distribution $\rho(x,p,t)$ involves a momentum function $p_x(x,t)$ which may depend on the state. For appropriate choices of p_x, the de Broglie–Bohm and Roy–Singh distributions can be obtained.

The distribution satisfies an equation of continuity, which follows from the standard quantum mechanical equation of continuity satisfied by ρ_x:

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= \frac{\partial\rho_x}{\partial t}\delta - \frac{\partial p_x}{\partial t}\rho_x\delta' \\
&= -\partial_x(v\rho_x)\delta - \frac{\partial p_x}{\partial t}\rho_x\delta' \\
&= -\partial_x(v\rho_x\delta) - \frac{\partial p_x}{\partial x}v\rho_x\delta' - \frac{\partial p_x}{\partial t}\rho_x\delta' \\
&= -\partial_x(v\rho_x\delta) - \frac{\partial}{\partial p}\left[\left(\frac{\partial p_x}{\partial t} + v\frac{\partial p_x}{\partial x}\right)\rho_x\delta\right].
\end{aligned} \tag{6}$$

The last line shows the equation of continuity in phase space for the joint distribution and involves the phase space velocity at (x, p_x). This velocity can be derived from a classical or causal Hamiltonian

$$H_c = \frac{(p - A(x,t))^2}{2m} + U(x,t), \tag{7}$$

where the vector and scalar potentials A, U have to satisfy the conditions

$$v = \frac{p_x - A}{m}, \tag{8}$$

$$\frac{\partial p_x}{\partial t} + v\frac{\partial p_x}{\partial x} = -U' + \frac{A'(p_x - A)}{m}. \tag{9}$$

The first line determines A in terms of v (which is known from standard quantum mechanics in terms of the quantum mechanical Hamiltonian entering the Schrödinger equation and the second line fixes U:

$$-U' = \frac{\partial p_x}{\partial t} + \left(\frac{1}{2}mv^2\right)'. \tag{10}$$

5. Joint probability distributions for two-timers

All this was for a single time. What happens when there are two times? The joint probability distribution can be introduced exactly as before. A p_x is required, and is to be chosen in the same manner as in the one-time case. It will depend on the wave-function and hence, in the present case, on the two time variables. Using p_x, a $\rho(x, p, t_a)$ is defined. It will satisfy an equation of continuity for each time, as can be checked by using the two equations of continuity for the x-probability distribution that follow from the two wave equations. The velocities and rates of change of momenta entering these equations of continuity are then used to determine causal Hamiltonians

$$H_{ca} = \frac{(p - A(x, t_b))^2}{2m_a} + U_a(x, t_b), \tag{11}$$

with

$$v_a = \frac{p_x - A}{m_a}, \tag{12}$$

$$-U_a' = \frac{\partial p_x}{\partial t_a} + \left(\frac{1}{2} m_a v_a^2\right)'. \tag{13}$$

It should be pointed out that this equation has been written for one space dimension: for three space dimensions the equation for U changes to

$$-\nabla U_a = \frac{\partial \vec{p}_x}{\partial t_a} + \nabla \left(\frac{1}{2} m_a v_a^2\right) - \vec{v}_a \times (\nabla \times \vec{p}_x). \tag{14}$$

All this is relatively straightforward, except that there is the question of consistency. The two causal Hamiltonians so obtained have to satisfy a relation like (2) above. Let us check whether it is indeed satisfied.

Now the two quantum mechanical Hamiltonians for the two times must be consistent. Assuming that both are of the form

$$H_a = \frac{p^2}{2m_a} + V_a(x), \tag{15}$$

one can see that they have to be proportional to each other:

$$H_1 = cH_2. \tag{16}$$

In principle there can also be a constant, but it can be absorbed. This simply means that

$$m_1 = \frac{1}{c} m_2, \quad V_1 = cV_2. \tag{17}$$

The relation between the Hamiltonians implies that the time-derivatives of the wave-function are related:

$$\frac{\partial}{\partial t_1} \approx c \frac{\partial}{\partial t_2} \text{ (when acting on } \psi). \tag{18}$$

Consequently,

$$\vec{v}_1 = c\vec{v}_2, \quad m_1\vec{v}_1 = m_2\vec{v}_2. \tag{19}$$

It is now clear from (13) that

$$U_1 = cU_2, \tag{20}$$

and consequently

$$H_{c1} = cH_{c2}. \tag{21}$$

This also means that

$$\{H_{c1}, H_{c2}\} = 0 \tag{22}$$

and

$$\frac{\partial H_{c1}}{\partial t_2} = \frac{\partial H_{c2}}{\partial t_1}. \tag{23}$$

Thus the consistency condition is satisfied: the causal Hamiltonians work.

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