PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 58, No. 2
February 2002
pp. 333–341

From chaos to disorder: Statistics of the eigenfunctions of microwave cavities

PRABHAKAR PRADHAN[1,2] and S SRIDHAR[1]
[1]Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA
[2]Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge MA 02139, USA

Abstract. We study the statistics of the experimental eigenfunctions of chaotic and disordered microwave billiards in terms of the moments of their spatial distributions, such as the inverse participation ratio (IPR) and density-density auto-correlation. A path from chaos to disorder is described in terms of increasing IPR. In the chaotic, ballistic limit, the data correspond well with universal results from random matrix theory. Deviations from universal distributions are observed due to disorder induced localization, and for the weakly disordered case the data are well-described by including finite conductance and mean free path contributions in the framework of nonlinear sigma models of supersymmetry.

Keywords. Quantum chaos; localization; mesoscopic physics; random matrix theory.

PACS Nos 73.23.-b; 05.45.Mt; 73.20.Dx; 73.20.Fz

1. Introduction

This paper briefly summarizes results and insights gained from experimental studies of eigenfunctions of chaotic and disordered 2D billiards. The experiments, which utilize microwave cavities, enable us to tune the degree of localization by varying two parameters, the frequency f and the mean free path l. Thus we are able to access the ballistic limit $l \gg \lambda$ (the wavelength) as well as the strongly localized limit $l \prec \lambda$. In our earlier work [1] we have experimentally shown that the spatial intensity distribution of an eigenfunction of a chaotic billiard quantitatively agrees with the random matrix theory (RMT). Recently, we have obtained further results on the statistics of the eigenfunctions [2] of chaotic and disordered systems, which are reviewed in this paper. The results of our experiments on disordered microwave cavities are quantitatively described by calculations based upon the supersymmetry sigma models carried out by Efetov [3], Prigodin and Altshuler [4], Mirlin [5] and others. In this paper we explore the eigenfunction statistics of chaotic and disordered systems using inverse participation ratio as a measure of the disorder strength.

2. Theoretical background: RMT and nonlinear sigma models

It was shown by Berry that the spatial amplitude distribution of the wave function of an electron in a chaotic cavity can be generated from the random superposition of plane waves.

This leads to the universal spatial distribution of the eigenfunction $P(\Psi)$ which is Gaussian, and which implies that the spatial intensity distribution of the eigenfunction $P(|\Psi|^2)$ is Porter–Thomas [6]. Spatial correlations are also obtained, such as the density-density autocorrelation $\langle|\Psi(r)|^2|\Psi(r')|^2\rangle = 1 + 2J_0^2(k|r-r'|)$ for a 2-D chaotic billiard.

The moments of the spatial integral of density of an eigenfunction $I_n = \int|\Psi(\bar{r})|^{2n}\mathrm{d}^3r$ are important measures of the localization, particularly the second moment, the inverse participation ratio I_2 (IPR), and its statistics $P_{I_2}(I_2)$, which describes important properties of the chaotic and disordered systems. The eigenfunctions of a classically chaotic billiard are delocalized wave functions in the limit of an infinite system. The moments I_n have fixed values with no fluctuations, with $I_2 = 3.0$ in 2-D, i.e. $P_{I_2}(I_2) = \delta(I_2 - 3.0)$ [4,5]. The universalities have been studied for some time within the theoretical framework of the RMT.

From the perspective of transport in disordered systems, the chaotic system can be considered as the ballistic limit corresponding to infinite mean free path and infinite conductance. Due to the presence of weak disorder, the universal properties break down, and system properties deviate from the universal values. A perturbative treatment has been achieved using nonlinear sigma models of supersymmetry, originally motivated by the problem of electrons in a disordered metal, which can be termed quantum diffusion [7]. The supersymmetry approach, which treats an electron in a potential $V_{\mathrm{imp}}(r)$ characterized by a Gaussian distribution of random disorder due to impurities, enables the calculation of spectral correlations, such as level spacing statistics, form factor, etc. and eigenfunction correlations including amplitude $P(\Psi)$ and density $P(|\Psi|^2)$ distributions, and spatial autocorrelations $\langle|\Psi(r)|^2|\Psi(r')|^2\rangle$. In the spatially homogeneous 0-mode limit of the theory, which corresponds to the ballistic limit $kl \gg 1$, the theory reduces to that of the chaotic limit and yields the results of RMT, such as the Porter–Thomas density distribution, characteristic of systems with a Gaussian distribution of amplitudes.

With increasing disorder, i.e. with a finite mean free path and finite conductance, the system statistics deviate from the RMT, and now the system can be described by 1D sigma models as a perturbative correction with a finite conductance g and a finite disorder controlled parameter kl, where k is the wave vector and l is the finite mean free path. The leading term, the 1-mode limit, yields the leading corrections due to incipient localization, with higher orders (in principle) leading to increasing localization. The intensity-intensity auto-correlation $\langle|\Psi(r)|^2|\Psi(r')|^2\rangle$ can also be described by sigma model calculations. For a disordered system, correlation is large at $r - r' = 0$ and dies out with increasing distance.

3. Experimental details

Recent developments in experimental techniques have made it possible to measure the eigenvalue and eigenfunction properties of thin cylindrical microwave cavities – these studies have enabled quantitative studies of issues in quantum chaos and localized media on laboratory length scales with very high precision experiments.

In 2D, Schrödinger and Maxwell equation map onto each other in the sense that the Helmholtz equation for the z-component of the electric field becomes $(\nabla^2 + k^2)\psi = 0$, where $\psi = E_z$. 2D cavities were made between two parallel copper plates, and the shape and size of a cavity was made according to the need. The height of the cavity is $d = 6$ mm, and the cavity is effectively 2D for the range of microwave wavelengths used in the experiments. The electric field distribution for the transverse magnetic (TM) modes of such a cavity obeys the Schrödinger wave equation in two dimensions [8–10].

For chaotic billiard experiments, the shape of the cavity was made like a Sinai stadium or like Sinai billiards. For disordered closed billiard experiments, the cavity has closed rectangular (44 cm $\times$ 21.8 cm) boundaries, and the disordered scattering centers were imposed by placing randomly circular 2D plates of 1 cm radius. The scattering mean free path was changed by changing the number of the scattering centers. Several realizations of a scattering system were done by changing the scattering positions with a random number generator and keeping the total number of scatterers the same. A variety of experiments on closed and open quantum chaotic systems have been carried out using such microwave cavities [8–10]. A particularly powerful aspect of the experiment is the ability to directly measure wave functions using cavity perturbation techniques [8].

Experiments on the eigenvalue statistics have enabled stringent tests of the eigenvalue properties of chaotic systems [9]. Experiments on disordered billiards show a very good realization of 2D disordered systems [1], and which motivated further theoretical works in this field [11,4,5]. Most recently, we have carried out a systematic study of the statistics of the experimental eigenfunctions for chaotic and disordered systems [2]. These papers show tests of several analytical results of RMT and the nonlinear model of supersymmetry which was developed mainly for disordered granular media and mesoscopic systems. At present, the eigenfunctions of mesoscopic systems are difficult to access for controlled experiments due to their smaller size, and consequently microwave cavity experiments play an unique and important role for wave function statistics.

4. Experimental eigenfunctions and their statistics

The utility of analysing eigenfunctions in terms of their IPR is discussed here for a few representative eigenfunctions. For convenience, we will use the notation $I_2 = \int |\Psi(r)|^4 \mathrm{d}v = \int u^2 \mathrm{d}v$, $u = |\Psi(r)|^2$, $w = (I_2 - 3)/6$, and $I_1 = \int |\Psi(r)|^2 \mathrm{d}v = 1$, where the integration is done over the 2D volume. More than 250 eigenfunctions are analysed for chaotic and disordered systems, and each eigenfunction has 3200 spatial data points.

The representative eigenfunctions of the disordered billiard with $N = 36$ scatterers have been shown in figures 1a, b, c and d. Figure 1a shows the eigenfunction of a disordered billiard with a low eigenfrequency $f = 3.84$ GHz has localized eigenstate with IPR value $I_2 = 11.22$. With increased frequency (energy), the same disordered medium shows more delocalized eigenstates with decreasing IPR values: Figure 1b, $I_2 = 9.64$ at 4.49 GHz, figure 1c, $I_2 = 7.52$ at 6.37 GHz, and figure 1d, IPR value $I_2 = 4.36$ at 7.20 GHz. This can be regarded as tuning the degree of localization by changing the frequency (energy). In fact, we have shown [2] that this tuning path follows a power law decay with an exponent $\frac{1}{2}$, consistent with theoretical calculations [12].

Figure 1e and f show representative eigenfunctions of a chaotic billiard, which are 'delocalized', with IPR $I_2 = 3.13$ at 6.05 GHz (figure 1e), and $I_2 = 3.02$ at 7.54 GHz (figure 1f). These eigenfunctions have their IPR values close to the universal value $\langle I_2 \rangle = 3$ in 2D.

A comparison of the experimental results of chaotic and localized eigenfunction statistics are shown in figure 2 for (i) spatial density distribution of eigenfunctions, (ii) statistics of eigenfunctions in terms of their IPR values, and (iii) intensity auto-correlation of the eigenfunctions.

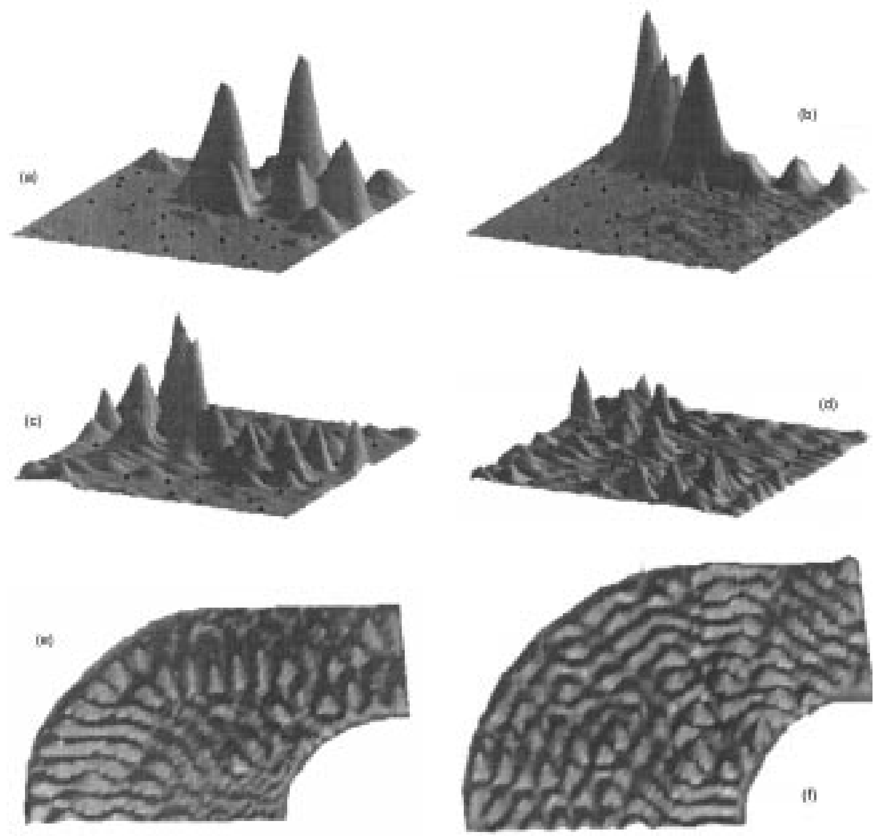

Figure 1. Representative eigenfunctions of a disordered billiard with number of scatterers is 36: (**a**) IPR value $I_2 = 11.22$ at $f = 3.84$ GHz, (**b**) $I_2 = 9.64$ at 4.49 GHz, (**c**) $I_2 = 7.52$ at 6.37 GHz and (**d**) $I_2 = 4.36$ at 7.20 GHz. The plots show that the eigenfunctions are more localized (larger IPR value) at a lower frequency and more delocalized (smaller IPR value) at a higher frequency. Representative eigenfunctions of a chaotic billiard: (**e**) $I_2 = 3.13$ at 6.05 GHz, and (**f**) $I_2 = 3.02$ at 7.54 GHz.

A. *Spatial density distribution of eigenfunctions*

Figure 2a shows that the spatial intensity distribution $P(|\Psi|^2)$ of eigenfunctions of a chaotic system obeys the Porter–Thomas distribution. The PT distribution is obtained from RMT, as well as random superposition of plane waves and the 0-D sigma model. The deviation from the PT distribution due to finite localization of a disordered medium is shown in figure 2b. This can be modeled by introducing finite mean free path in the problem and a quantitative description of the data has been achieved [11].

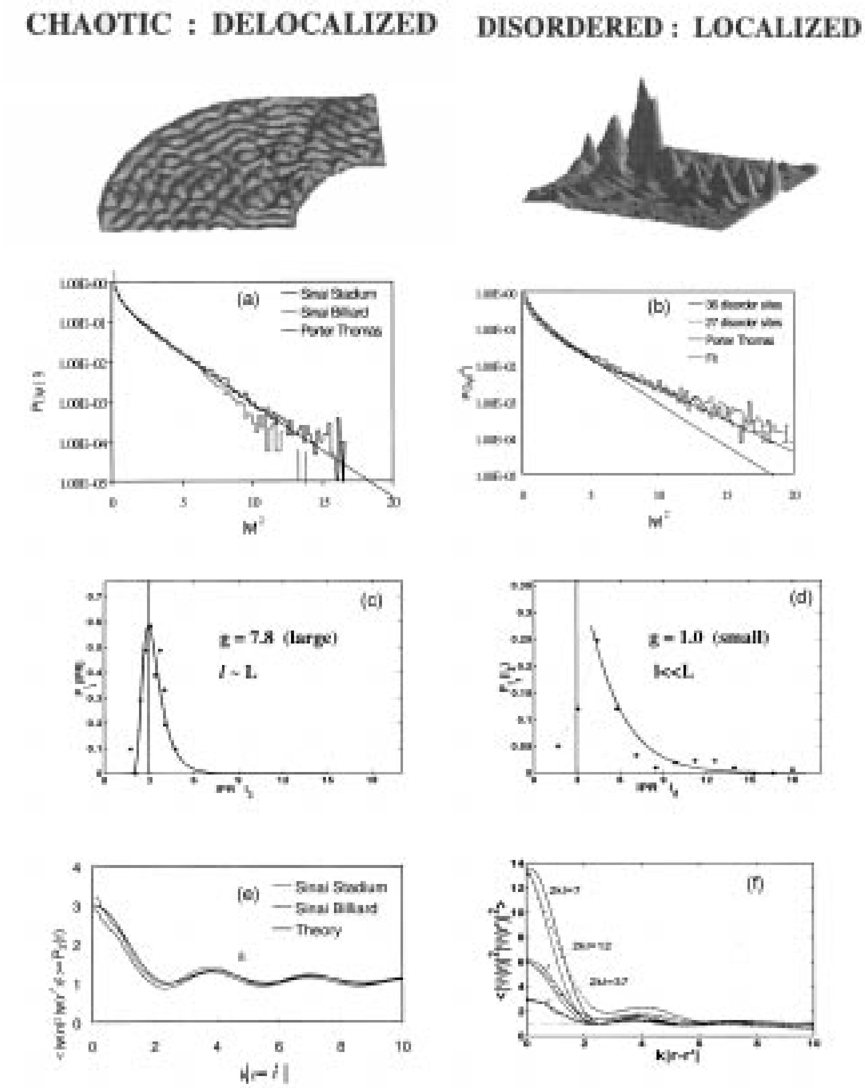

Figure 2. The spatial intensity distribution of experimental eigenfunctions: (**a**) Porter–Thomas distribution for chaotic cavity, and (**b**) non Porter–Thomas distribution for disordered cavity. (**c**) The IPR $P_{I_2}(I_2)$ distribution of the chaotic Sinai-stadium billiard is nearly symmetric around the universal mean value 3.0. (**d**) The IPR distribution $P_{I_2}(I_2)$ of the disordered billiards is strongly asymmetric and non-Gaussian. The lines represent calculations based on the nonlinear sigma-models. Intensity auto-correlation $\langle \Psi^2(r)\Psi^2(r') \rangle$ of eigenfunctions of (**e**) Sinai-stadium billiards, and (**f**) disordered billiards with different fixed disordered strengths $2kl$, experiment (dotted lines), eq. (4) (dashed lines), and numerical solution of eq. (4) with $2kl$ =13 and 7 (solid lines).

B. *Statistics of IPR of eigenfunctions*

The distribution $P_{I_2}(I_2)$ of the IPR I_2 for the chaotic billiards is shown in the figure 2c. The distribution is nearly symmetric around $\langle I_2 \rangle = 3$. RMT calculations indicate that there should be no fluctuations, but this only applies for an infinite system. The finite width of the distribution is due to the boundary scattering, and can be quantitatively described by large but finite $2kl$ and g values [13]. A quantitative description is given later.

In contrast the IPR distribution (figure 2d) for a disordered billiard with mean free path $l = 5.1$ cm ($N = 36$ scatterers) is strongly asymmetric and has a mean value much larger than the universal value of 3.0. These data can be modeled with a finite conductance g. A dimensionless conductance can be defined as $g = \ln(R/l)/\langle w \rangle$, where R is the system size and l is the mean free path and $\langle .. \rangle$ is the realization average for a fixed 'disordered strength' $2kl$.

Recent theoretical calculations by Prigodin and Altschuler [4] have shown that when $I_2 \gg \langle I_2 \rangle$, the IPR distribution follows an exponential decay law

$$P_{I_2}(I_2) = C_2 \sqrt{\frac{g}{I_2}} \exp\left(-\frac{\pi}{6} g I_2\right) \tag{1}$$

while $P(I_2)$ for $I_2 < \langle I_2 \rangle$ is [4]

$$P_{I_2}(I_2) = C_1 \frac{g}{2} \exp\left[-\frac{g}{6}(I_2 - \langle I_2 \rangle) - \frac{\pi}{2} \mathrm{e}^{-\frac{g}{3}(I_2 - \langle I_2 \rangle)}\right], \tag{2}$$

where C_1 and C_2 are the normalization constants. For the sample with finite mean free path $l = 5.1$ cm, and assuming random phase approximation within a small window where $2kl$ is fixed and independent of IPR distribution [18], experimental data matches well with eq. (1) with a conductance value $g = 1.0$ as shown in figure 2d. The distribution is also asymmetric beyond the universal average value $\langle I_2 \rangle = 3$, for the chaotic billiards.

For the chaotic billiard, the data matches well with eq. (2), as shown in figure 2c for the finite conductance $g = 7.8$ and $2kl = 37$, quite large values that we can consider the system as infinite, i.e., the length scale is same as the system size. Values of g are consistent with the parameters of the experimental microwave cavity.

C. *Intensity-intensity auto-correlation of the eigenfunctions*

In an earlier work we have shown that the intensity-intensity auto-correlations for the chaotic cavity has a universal behavior associated with Friedel oscillations [19]. We have extended the density auto-correlation calculation from chaotic to localized eigenfunctions, and analysed the results analytically and numerically. The intensity-intensity auto-correlation for an arbitrary disordered strength $2kl$ can be calculated as follows: Let us define $K(r) = |\mathrm{Im}\, G(r')|^2/(\pi\nu)^2$, where $G(|r - r'|) = \langle r | (E - H)^{-1} | r' \rangle$ is the Green function of the disordered system Hamiltonian, then one can show that [20]

$$K(r) = \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + y^2} J_0 \left[kr \left(1 + \frac{1}{2kl} y \right) \right] \mathrm{d}y \right|^2,$$

where J_0 is the zeroth order Bessel function.

For chaotic billiards the auto-correlation follows an analytical form

$$\left\langle \Psi^2(r)\Psi^2(r')\right\rangle = 1 + (I_2 - 1)J_0^2(r - r'). \tag{3}$$

This also corresponds to that for a flat disordered potential with Gaussian fluctuations, when $2kl \gg 1$. We extend the previous calculations for the finite localization case. Repeating the earlier calculations of [19] with a finite kl value, it can be easily shown that, the expression for the auto correlation can be approximately expressed in a region $r - r' \precsim l$ by a decay length scale of scattering mean free path length l as follows:

$$\left\langle \Psi^2(r)\Psi^2(r')\right\rangle \simeq 1 + (I_2 - 1)K(k|r - r'|) \tag{4}$$

$$\simeq 1 + (I_2 - 1)J_0^2(k|r - r'|)\mathrm{e}^{-\frac{k|r-r'|}{kl}}. \tag{5}$$

With kl large, the above expression, eqs (4) and (5), will converge to the expression of a chaotic system, eq. (3).

We have plotted the value of $\left\langle \Psi^2(r)\Psi^2(r')\right\rangle$ for different types of chaotic billiards in figure 2e and it agrees well with eq. (3).

For the disordered billiards, K was solved numerically, and also approximate analytical results have been plotted in figure 2f, and were compared with the results of $\left\langle \Psi^2(r)\Psi^2(r')\right\rangle$ calculated from experimental data. The agreement is quantitatively excellent in the described region. For Sinai billiards, $2kl = 37$ is a very large number and described well for a Gaussian distribution of the wave functions (figure 2f). For the disordered billiards, the matchings are shown for the $2kl = 7$ and 3 (figure 2f). The correlation is large at $r - r' = 0$, i.e. equals $\left\langle \Psi^4(r)\right\rangle = I_2$, and the correlation decays to zero with increasing spatial distance, as the auto-correlation is negligible at a larger distance beyond the scale of the localization length.

5. Conclusions

In this paper we have reviewed statistical properties of wave functions of chaotic and disordered systems using the second moment of the spatially integrated intensity of the eigenfunctions. We have shown that the statistics agrees well with random matrix theory for chaotic systems, and with the leading perturbative correction of the nonlinear sigma model of super symmetry for incipient disordered media. The perspective that emerges from the microwave experiments is summarized in figure 2 in terms of spatial intensity distribution (figure 2a, b), statistics of inverse participation ratio (IPR) of eigenfunctions (figure 2c, d), and intensity-intensity auto-correlation function (figure 2e, f). The universal (2D) results of quantum chaos, characterized by Gaussian fluctuations of Ψ, are achieved in chaotic billiards, albeit with small corrections due to correlations induced by boundary scattering. The importance of correlations due to coherent interference between scattering events are amplified in the model disordered billiards. By tuning kl we can observe a path from the extended ($I_2 \sim 4$) states to strongly localized ($I_2 \sim 20$) states, as can be seen in a I_2 vs. f^2 plot (figure 2 in ref. [2]). This is similar to tuning from a metal to insulator by increasing energy! This is also accompanied by deviations from the Porter–Thomas (PT) distributions, and strong level-to-level fluctuations, leading to a strongly asymmetric distribution $P_{I_2}(I_2)$, and strong spatial decay of correlations, all indicating non-Gaussian amplitude distributions. For not too strongly localized states, we have shown agreement with the supersymmetry theories, parameterized by finite conductivity g (figures 2b, d and f). The conductivity g depends monotonically on kl, with the ballistic chaotic limit corresponding to $g, kl \to \infty$. A systematic study of wave function statistics is under way to explore more strongly localized states for which however there is no available theory currently.

The eigenvalue and eigenfunction statistics of a disordered medium for finite g have been calculated [4] by solving a Perron–Frobenius operator of a classical diffusion equation, whose (real) eigenvalues are related to the poles of a classical zeta function. Recently, a direct experimental observation of the complex poles of the Liouville operator for an open chaotic system, the so-called classical Ruelle–Pollicot resonances, have been observed by us in microwave experiments on open n-disk systems [14–17]. This latter work thus provides a physical reality to these eigenvalues in open systems which have played a significant role in the theory of the disordered systems, and provides a close connection between classical and quantum diffusion.

While the nonlinear sigma model provides a useful, even quantitative, framework to discuss arising from coherent interference leading to localization, its perturbative approach limits it to the leading corrections to the ballistic or RMT limit. The experiments are not limited to the regime of large g i.e. $kl \gg 1$, but can easily reach the strong localization limit $kl \prec 1$. This regime requires higher order calculations beyond the 1-mode that are quite formidable. Other approaches to this problem deserve attention. One possibility that intrigues us is the information theoretic approach based upon maximum entropy employed by Kumar and collaborators [21]. While this has thus far been applied only to quasi 1D, extensions to higher dimensions would be worth pursuing.

Acknowledgement

It is a pleasure to dedicate this paper to Professor Narendra Kumar. PP had the honor of carrying out his Ph.D. thesis research under Professor Kumar's guidance. SS has had the pleasure of numerous discussions on physics that have always been enthralling and memorable. This work was supported by NSF-PHY-0098801.

References

[1] A Kudrolli, V Kidambi and S Sridhar, *Phys. Rev. Lett.* **75**, 822 (1995)
[2] P Pradhan and S Sridhar, *Phys. Rev. Lett.* **85**, 2360 (2000)
[3] K B Efetov, *Supersymmetry in disorder and chaos* (Cambridge University Press, 1997)
[4] V Prigodin and B Altshuler, *Phys. Rev. Lett.* **80**, 1944 (1998)
[5] A D Mirlin, *Phys. Rep.* **326**, 259 (2000)
[6] M Srednicki, *Phys. Rev.* **E54**, 954 (1996)
[7] E Akkermans and G Montambaux , LANL e-print, cond-mat/0104013
[8] (a) S Sridhar, *Phys. Rev. Lett.* **67**, 785 (1991)
(b) S Sridhar, D Hogenboom and B A Willemsen, *J. Stat. Phys.* **68**, 239 (1992)
[9] A Kudrolli, S Sridhar, A Pandey and R Ramaswamy, *Phys. Rev.* **E49**, R11 (1994)
[10] (a) M Barth, U Kuhl and H-J Stöckmann, *Phys. Rev. Lett.* **82**, 2026 (1999)
(b) H Alt *et al*, *Phys. Rev. Lett.* **74**, 62 (1995)
(c) L Sirko, P Koch and R Blumel, *Phys. Rev. Lett.* **78**, 2940 (1997)

[11] V Fal'ko and K Efetov, *Phys. Rev.* **B50**, 11267 (1994)
[12] Y V Fyodorov and A D Mirlin, *Phys. Rev.* **B55**, 16001 (1997)
[13] B A Muzykantskii and D E Khmelnitskii, *Phys. Rev.* **B51**, 5480 (1995)
[14] K Pance, W T Lu and S Sridhar, *Phys. Rev. Lett.* **85**, 2737 (2000)
[15] W T Lu, K Pance, M Rose and S Sridhar, *Phys. Rev. Lett.* **82**, 5233 (1999)
[16] W T Lu, K Pance, P Pradhan and S Sridhar, *Phys. Scr.* **T90**, 238 (2001)
[17] P Gaspard, *Chaos* **3(4)**, 427 (1993)
[18] P Pradhan, LANL e-print, cond-mat/9703255
[19] V Prigodin, N Taniguchi, A Kudrolli, V Kidambi and S Sridhar, *Phys. Rev. Lett.* **75**, 2392 (1995)
[20] V N Prigodin, B L Altshuler, K E Efetov and S Iida, *Phys. Rev. Lett.* **72**, 546 (1994)
[21] P A Mello, P Pereyra and N Kumar, *Ann. Phys. (N.Y.)* **181**, 209 (1988)