PRAMANA
— journal of
physics

Vol. 53, No. 2
August 1999
pp. 253–269

Genericity and stability of naked singularities arising in an inhomogeneous dust collapse

S H GHATE, R V SARAYKAR and K D PATIL
Department of Mathematics, Nagpur University Campus, Nagpur 440 010, India

MS received 22 September 1998; revised 19 April 1999

Abstract. In this paper, we consider an inhomogeneous dust collapse, and extend earlier works of Jhingan, Joshi, and Singh to the case where initial density and velocity distributions are finitely differentiable functions of co-moving coordinate r. We study the occurrence of naked singularities under various conditions on the derivatives of initial density and velocity distributions in marginally as well as non-marginally bound case. We then study their stability and genericity with respect to perturbations in the initial data in an appropriate topological sense.

Keywords. Generic properties; stability; naked singularity; gravitational collapse.

PACS Nos 04.20.Dw; 04.20.D; 04.25; 04.40; 04.70.B; 95.30.Sf

1. Introduction

Over the last 15 years or so classical relativists interested in formulating a proper provable version of cosmic censorship are studying existence of naked singularities seriously. This is essential because a range of physical parameters like density, velocity, pressure which lead to naked singularities (in certain models of gravitational collapse) is to be specified and deleted in formulating this conjecture. Naked singularities are known to exist in the gravitational collapse of Vaidya space-times, inhomogeneous Tolman–Bondi space-times, and space-times containing matter such as perfect fluid. They are classified as, weak or strong, in different ways by different workers [1,2]. Shell-crossing singularities arising in Tolman–Bondi collapse are generally regarded to be gravitationally weak through which the space-time may be continued. The shell focussing singularities, however, can be weak or strong [1–3].

Singh and Joshi [4] have shown that in the case of spherical inhomogeneous dust collapse, naked singularities arise from regular initial data consisting of densities and velocities, which are expressible as power series in terms of co-moving coordinate r. Similar results are, also, expected in the general case of collapse [5–7]. An important question about naked singularities is, whether they are stable with respect to perturbation of initial data from which they arise. Another question is, how 'big' is the set of initial data, which causes these naked singularities.

Following [4], we study spherical inhomogeneous dust collapse, and show that an initial data of sufficiently differentiable density and velocity functions of co-moving coordinate r leads to naked shell focussing central singularity under certain conditions. We study marginally bound as well as non-marginally bound case. We first define what we mean by stability and genericity. We then formulate appropriate function spaces of densities and velocities, which are metric spaces under suitable metric, and then discuss stability as well as genericity with respect to the initial data set.

Thus in §2, we give background material in brief about Tolman–Bondi metric [8,9] describing spherically symmetric inhomogeneous dust collapse. In §3, we give precise definitions of stability and genericity following the theory of dynamical systems [10]. In §4, we consider marginally bound case $f(r) = 0$. We derive the conditions on derivatives of initial density distributions that lead the collapse to a naked singularity or otherwise. We also discuss stability and genericity of these singularities with respect to initial data set in the light of definitions given in §3. We also comment on the anomalous cases where occurrence of black holes is unstable with respect to perturbations in the initial data. In §5, we study all these properties for non-marginally bound case where $f(r) \neq 0$, i.e. initial data comprises of both density and velocity distributions. We also illustrate, with the help of examples, a case where density distribution is homogeneous, but still the presence of inhomogeneous velocity distribution leads the collapse to a naked singularity. We conclude the paper by comparing our work with other recent work on stability and comment on further work in progress in the case of spherical symmetric collapse for a general form of matter.

2. Background material

A spherically symmetric inhomogeneous dust collapse is given by the Tolman–Bondi metric in co-moving co-ordinates as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{R'^2}{1+f}\mathrm{d}r^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{1}$$

and the field equations are given by

$$T^{ij} = \epsilon\delta^i_t\delta^j_t, \quad \epsilon(t,r) = \frac{F'}{R^2R'}, \quad \dot{R}^2 = \frac{F}{R} + f, \tag{2–4}$$

where T^{ij} is the energy-momentum tensor, ϵ is the total energy density. F and f are arbitrary functions of r. The dot denotes derivative with respect to time, while the prime denotes derivative with respect to r. We consider a collapsing model and, hence, time derivative of R is negative in (4). This equation on integration gives

$$t - t_0(R) = \frac{-R^{3/2}G(-fR/F)}{\sqrt{F}}, \tag{5}$$

where G is given as

$$G(x) = \left\{ \begin{array}{ll} \frac{\sin^{-1}\sqrt{x}}{x^{3/2}} - \frac{\sqrt{1-x}}{x}, & 0 < x \leq 1 \\ 2/3, & x = 0 \\ \frac{-\sinh^{-1}\sqrt{-x}}{(-x)^{3/2}} - \frac{\sqrt{1-x}}{x}, & -\infty \leq x < 0 \end{array} \right\}. \tag{6}$$

Using the scaling freedom, we can choose

$$R(0,r) = r. \tag{7}$$

From eq. (5), we then get

$$t_0(r) = \frac{r^{3/2}G(-fr/F)}{\sqrt{F}}, \tag{8}$$

which gives the time at which the physical radius of the shell labeled by r becomes zero and the shell becomes singular.

The $F(r)$ is interpreted as the mass function for the cloud, and it is related to the density $\rho(r)$ of dust matter at the onset of collapse by

$$\frac{F'}{r^2} = \epsilon(0,t) \equiv \rho(r). \tag{9}$$

The term $f(r)$ is known as the energy function for the cloud, and is related to the initial radial velocity through eq. (4).

We introduce notations

$$X = \frac{R}{r^\alpha}, \quad \eta = \frac{rF'}{F}, \quad \beta = \frac{rf'}{f}, \quad p = \frac{rf}{F}, \quad P = pr^{\alpha-1}, \quad \lambda = \frac{F}{r^\alpha},$$
$$\Theta = \frac{t_0'\sqrt{\lambda}}{r^{\alpha-1}} = \frac{1+\beta-\eta}{(1+p)^{1/2}r^{3(\alpha-1)/2}} + \left(\eta - \frac{3}{2}\beta\right) G(-p)/r^{3(\alpha-1)/2}. \tag{10}$$

We get from these equations,

$$R' = r^{(\alpha-1)} \left[(\eta-\beta)X + \left[\Theta - \left(\eta - \frac{3}{2}\beta\right) X^{3/2}G(-PX)\right] \left[P + \frac{1}{X}\right]^{1/2}\right]$$
$$\equiv r^{\alpha-1}H(X,r). \tag{11}$$

The factor r^α, facilitates the examination of the structure of singularity. The exact value of $\alpha(>1)$ is to be chosen such that

$$\lim_{r\to 0} \Theta$$

is a nonzero finite value, and it will be different for different models.

If there are future radial null geodesics coming out of the singularity, with a well defined tangent at the singularity, then the quantity R' must tend to a finite limit in the limit of approach to the singularity in the past along these trajectories. The geodesic equation for the radial null geodesics is [4]

$$\frac{\mathrm{d}R}{\mathrm{d}u} = \left[1 - \frac{\sqrt{f+\lambda/X}}{\sqrt{1+f}}\right] \frac{H(X,u)}{\alpha} \equiv U(X,u). \tag{12}$$

If the outgoing radial null geodesics are to terminate in the past at the singularity at $r=0$ which occurs at time $t = t_0(0)$ at which $R(t_0(0),0) = 0$, then along these geodesics we have $R \to 0$ as $r \to 0$. If the null geodesics meet the singularity with a definite value of the tangent, then we get

$$X_0 = \lim_{\substack{R \to 0 \\ u \to 0}} \frac{R}{u} = \lim_{\substack{R \to 0 \\ u \to 0}} \frac{\mathrm{d}R}{\mathrm{d}u} = \lim_{\substack{R \to 0 \\ u \to 0}} U(X, u) = U(X_0, 0). \tag{13}$$

Since for an outgoing null geodesic $\mathrm{d}R/\mathrm{d}u$ must be positive, it follows that if a real and positive value of X_0 satisfies (13), the singularity will be naked. If no such real positive root exists, the singularity cannot be naked and the collapse may end into a black hole. In §4 and 5, we shall characterize the formation or otherwise of a naked singularity in terms of initial density and velocity functions, and their derivatives.

We note from eq. (9) that

$$F(r) = \int_0^r s^2 \rho(s) \mathrm{d}s + \text{ constant},$$

where the constant of integration can be taken to be zero (F is twice the mass inside the sphere of radius r).

Integrating by parts a number of times we get,

$$F(r) = \frac{r^3}{3}\rho - \frac{r^4}{12}\rho' + \frac{r^5}{60}\rho'' - \frac{r^6}{360}\rho''' + \int_0^r \frac{\rho''''}{360} s^6 \mathrm{d}s. \tag{14}$$

It follows that

$$\eta_0 \equiv \lim_{r \to 0} \eta(r) = \lim_{r \to 0} \frac{rF'}{F} = 3. \tag{15}$$

3. Stability and genericity of naked singularity

Here we give the appropriate definitions of stability and genericity. Our definitions of stability and genericity are based on the definitions of structural stability of a dynamical system, and genericity of a property of a dynamical system. We recall these definitions for the sake of completeness. For more details we refer the reader to chapter 7, §3, 4 of Abraham and Marsden [10].

Let M be a manifold on which a vector field or a dynamical system is defined. Let $\chi(M)$ be the space of all vector fields on M. We give Whitney C^r-topology on $\chi(M)$, generated by the norm

$$||f||_r = \sup \left\{ \sum_{k=0}^{r} ||D^k f(u)|| / u \in U \right\},$$

where E and F are vector spaces, U an open subset of E, and $f : U \to F \cdot D^k f$ denotes kth Frechet derivative of $F \cdot \chi(M)$ endowed with Whitney C^r-topology is denoted by $\chi^r(M)$.

A property of vector fields in $\chi^r(M)$ is a proposition $P(x)$ with a variable $x \in \chi^r(M)$. A property $P(x)$ with a variable $x \in \chi^r(M)$ is generic if the subspace $\{x \in \chi^r(M)/P(x)\} \subset \chi^r(M)$ contains a residual set.

A subset A of a topological space X is called residual if and only if A is the intersection of a countable family of open dense subsets of X. A topological space X is a Baire space if and only if every residual set is dense. We also know that every complete metric space and in particular every Banach space is a Baire space. Also, whether M is compact or not, $\chi^r(M)$ is a Baire space.

Let X be a vector field or a dynamical system on M. Then X is structurally stable if there is a neighborhood Φ of $X \in \chi^r(M)$ in the Whitney C^r-topology such that $Y \in \Phi$ implies X and Y are topologically conjugate i.e. they have equivalent phase portraits. This means that there is a homeomorphism $h : M \to M$ carrying oriented orbits of X to oriented orbits of Y.

Using these definitions analogously in our case to the evolution of initial data into a gravitational collapse leading to a naked or a covered singularity, we treat evolving initial data as a vector field or a dynamical system. In fact, the equations of radial null geodesics can be written as a plane autonomous system. Continuing our analogy, we consider the space of all initial data with sufficient differentiability defined on a collapsing compact spherical shell of dust, in place of $\chi(M)$, endowed with a suitable C^r-topology. Property P of a dynamical system becomes the property of initial data, namely, whether this initial data leads the collapse to a naked singularity or a black hole. Thus, the definitions of stability of a naked singularity and genericity of its occurrence can be stated as follows.

Let I_0 be the initial data set, which when evolves, leads the collapse to a naked singularity. We say that a naked singularity is stable, if there is a neighborhood I of I_0 in C^r-topology such that, if I_1 is another initial data in I, then I_1 also leads the collapse to a naked singularity. In other words, if the set of initial data leading the collapse to a naked singularity forms an open subset of C^r-space of all initial data, then the naked singularity will be stable.

Similarly, occurrence of a naked singularity will be said to be generic, if the set of all initial data leading to naked singularity, is a dense subset of the parent C^r-space.

In the light of above definitions, we now investigate stability and genericity of a naked singularity arising in marginally bound as well as non-marginally bound cases.

4. Marginally bound case

In this case $f = 0$, and hence $\beta = 0 = p$ identically (see §2). The space, under consideration in this case, will be the space D of all possible density distributions given as

$$D = \{\rho \in C^4[0, M], \rho(0) > 0, \rho(r) \geq 0\}$$

and on this set D, we define the metric

$$d(\rho, \rho_1) = \max_i \sup_{0 \leq r \leq M} |\rho^{(i)}(r) - \rho_1^{(i)}(r)|, \quad i = 0, 1, 2, 3, 4,$$

where, for convenience, the ith derivative of ρ is denoted by $\rho^{(i)}$. In the above definitions we restrict to derivatives up to order four as only they are involved in the case under consideration. We are considering the cloud to be described by r varying in $[0, M]$ at the start of collapse. With this metric, the maps $\rho \to \rho^{(i)}(0), i = 0, 1, 2, 3, 4$ are all continuous. This can be proved by using elementary topological arguments.

We now consider different density distributions and find conditions so that the central singularity of collapse is naked or otherwise. Using expressions (10) and (14) we get

$$\Theta(r) = \frac{\frac{-r^4}{4}\rho' + \frac{r^5}{20}\rho'' - \frac{r^6}{120}\rho''' + \int_0^r \frac{\rho'''' s^6}{120}\mathrm{d}s}{r^{3(\alpha-1)/2}\left[r^3\rho - \frac{r^4}{4}\rho' + \frac{r^5}{20}\rho'' - \frac{r^6}{120}\rho''' + \int_0^r \frac{\rho'''' s^6}{120}\mathrm{d}s\right]}. \tag{16}$$

Depending on the properties of the derivatives of ρ at $r = 0$ we get the following cases:

Case 1: $\rho'(0) \neq 0$. Choosing $3(\alpha - 1)/2 + 3 = 4$ i.e. $\alpha = 5/3$, we get from (16) that

$$\Theta_0 \equiv \lim_{r\to 0} \Theta = -\frac{\rho'(0)}{4\rho(0)}. \tag{17}$$

Also

$$\lambda_0 \equiv \lim_{r\to 0} \lambda = 0.$$

From eq. (13) we then get

$$X_0^{3/2} = -\frac{3\rho'(0)}{8\rho(0)}. \tag{18}$$

This equation will have positive root if and only if $\rho'(0) < 0$, and in this case the central singularity of collapse will be naked.

The set $D_1 = \{\rho \in D/\rho'(0) < 0\}$ is open in D. Hence, following our discussion in §3, we conclude that the naked singularity arising in the first case is stable. It is also obvious that D_1 is not dense in D.

In the case $\rho'(0) > 0$, as remarked in [11], we need not necessarily get a black hole as in this case there may be a shell crossing singularity (or singularities) at $r > 0$. Following [12], shell crossings are avoided if and only if $R' > 0$ and this condition can be translated in terms of density ρ as

$$\left(\frac{\rho(r)}{3} - r\frac{\rho'(r)}{12} + \frac{r^2\rho''(r)}{60}\cdots\right)\left(\frac{R}{r}\right)^{3/2} > \frac{r}{3}\left(\frac{\rho'(r)}{4} - \frac{r\rho''(r)}{20} + \cdots\right)\left(1 - \left(\frac{R}{r}\right)^{3/2}\right)$$

using the expression (14). In the above inequality, we have not written the full expression for the sake of brevity. Note that the set of density functions satisfying this condition is an open set due to the continuity of the functions involved (the set will be the inverse image of an open interval under composition of continuous functions). Thus, in case $\rho'(0) \neq 0$, the central singularity is covered if and only if the density ρ is such that $\rho'(0) > 0$, and it satisfies the above condition of avoidance of shell crossings. The required set being intersection of two open subsets of D, is itself open. We thus conclude that the black holes arising in this way are also stable in much the same way as the naked singularity. Due to non-denseness of the respective sets, it is clear that the set of density functions giving rise to both the naked singularity and black hole are non-generic.

Case 2: $\rho'(0) = 0, \rho''(0) \neq 0$. In this case, choosing $\alpha = 7/3$, we get

$$\Theta_0 = -\frac{\rho''(0)}{5\rho(0)} \tag{19}$$

and further that

$$\lambda_0 = 0.$$

Equation (13) then becomes

$$X_0^{3/2} = -\frac{3\rho''(0)}{20\rho(0)} \tag{20}$$

and this equation has a positive root if and only if $\rho''(0) < 0$ and in this case the singularity is naked. To discuss the stability of the naked singularity arising in this case, consider the subset D_2 of D given as

$$D_2 = \{\rho(r) \in D \ni \rho'(0) = 0, \rho''(0) < 0\}.$$

It can be shown that the set D_2 is not open. For let $\epsilon > 0$ be arbitrary and let $\rho \in D_2$. Define ρ_1, ρ_2 as

$$\rho_i(r) = \rho(r) + (-1)^i \frac{\epsilon}{2(M+1)} r, \quad i = 1, 2.$$

Then, $\rho_1'(0) < 0$, $\rho_2'(0) > 0$ and $\rho_1''(0) = \rho_2''(0) = \rho''(0)$. Without loss of generality, we can choose ϵ small enough so that

$$\frac{\epsilon M}{2(M+1)} < \min_{0 \leq r \leq M} \rho(r).$$

This will ensure that both ρ_1 and ρ_2 are in D. We also have

$$\begin{aligned} d(\rho, \rho_i) &= \sum_{j=0}^{3} \sup_{0 \leq r \leq M} |\rho^{(j)} - \rho_i^{(j)}| \\ &= \sup_{0 \leq r \leq M} \left| \frac{\epsilon r}{2(M+1)} \right| + \sup_{0 \leq r \leq M} \left(\left| \frac{\epsilon}{2(M+1)} \right| \right) \\ &= \frac{\epsilon M}{2(M+1)} + \frac{\epsilon}{2(M+1)} = \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Thus, every neighborhood of ρ contains ρ_1, and ρ_2 such that none of them is in D_2. While the density ρ_1 as initial density leads the collapse to central naked singularity, the density ρ_2 leads either to shell crossings or black hole. We therefore conclude that the naked central singularity forming in this case is unstable. Same argument will also hold true for the black holes forming in the sub case of $\rho'(0) = 0, \rho''(0) > 0$. This is an anomalous case where occurrence of black holes is unstable.

As the closure of the set D_2 is given by

$$\{\rho(r) \ni \rho'(0) = 0, \rho''(0) \leq 0\}$$

which obviously is not the whole of D. It is clear that the above initial data of density functions giving rise to central naked singularity is non-generic.

Case 3: $\rho'(0) = \rho''(0) = 0, \rho'''(0) \neq 0$. We choose $\alpha = 3$ which gives

$$\Theta_0 = -\frac{\rho'''(0)}{12\rho(0)}, \quad \lambda_0 = \frac{\rho(0)}{3}. \tag{21}$$

Equation (13) then becomes

$$X_0 - \frac{1}{3}\left\{1 - \sqrt{\frac{\rho(0)}{3X_0}}\right\}\left\{X_0 + \frac{1}{\sqrt{X_0}}\left(-\frac{\rho'''(0)}{12\rho(0)}\right)\right\} = 0$$

i.e.

$$2X_0^2 + \sqrt{\frac{\rho(0)}{3}}X_0^{3/2} + \frac{\rho'''(0)}{12\rho(0)}\sqrt{X_0} - \frac{\rho'''(0)}{12\sqrt{3\rho(0)}} = 0. \tag{22}$$

Let

$$x = \sqrt{\frac{3X_0}{\rho(0)}}.$$

Substituting in (20) we get on further simplification

$$2x^4 + x^3 + \xi x - \xi = 0, \tag{23}$$

where

$$\xi = \frac{\sqrt{3}\rho'''(0)}{4(\rho(0))^{5/2}}.$$

The quartic equation (23) has positive roots if and only if $\xi < \xi_1$ with $\xi_1 = -25.9904$ i.e.

$$\frac{\sqrt{3}\rho'''(0)}{4(\rho(0))^{5/2}} < -25.9904 \tag{24}$$

and in this case the central singularity will be naked. Since in this case we have $\rho'(0) = \rho''(0) = 0$, for any $\epsilon > 0$, we can find ρ_1, ρ_2 in D defined as

$$\rho_i(r) = \rho(r) + (-1)^i \frac{\epsilon}{2(M^2 + 2M + 2)} r^2, \quad i = 1, 2.$$

It follows that ρ_1, ρ_2 do not belong to the class studied in this case although $d(\rho, \rho(i)) < \epsilon$, $i = 1, 2$. Giving arguments similar to the previous case, we thus conclude that the naked singularity arising in this case is unstable and the set of initial data under consideration is non-generic. Similar argument holds for the set of density functions $\rho'(0) = \rho''(0) = 0$ and

$$\frac{\sqrt{3}\rho'''(0)}{4(\rho(0))^{5/2}} \geq -25.9904.$$

This is also an anomalous case where occurrence of black holes is unstable under perturbations of initial data.

Case 4: $\rho'(0) = \rho''(0) = \rho'''(0) = 0$. In this case we have to choose $\alpha >= 11/3$.

For such value of $\alpha, \lambda_0 = \infty$ and equation (13) does not have any finite solution and thus the collapse ends into a black hole. Due to the vanishing of the first three derivatives of density at $r = 0$, it is clear, using the arguments similar to the previous cases that the set of such densities can form neither an open nor a dense subset of the set D of all possible density distributions. The black holes, forming from such initial density distributions, are thus, not stable, neither is their occurrence generic.

The union of all the density distributions that give rise to a central naked singularity in the three cases studied above is also not dense. Hence, we conclude that the naked central singularities arising from the evolution of continuously differentiable initial density distributions in a marginally bound collapse are non-generic.

The anomalous situation of black holes being unstable is a peculiar feature of the collapse studied here, as the nature of singularity of collapse depends sensitively on the signs of the derivatives of initial density functions. Similar situation occurs for Oppenheimer–Snyder model that describes homogeneous dust collapse leading to the formation of a black hole. As remarked in [13], if the O–S initial data is perturbed by switching on an infinitesimal negative $\rho'(0)$ or $\rho''(0)$ terms, we get a naked singularity instead. The black hole is not stable to small perturbations. On the other hand, if the O–S black hole scenario is perturbed by switching on a small $\rho'''(0)$, it continues to be a black hole. Only large enough perturbations from homogeneity at the level of the third derivative convert the black hole to a naked singularity.

Genericity considered here is in a broad sense of denseness, though the set of all initial density functions which leads the collapse to a naked singularity or otherwise is a substantially big subset of the parent set.

We now study non-marginally bound case and obtain similar results.

5. Non-marginally bound case

Let v denote initial velocity profile over the cloud. From eq. (4) we then get

$$v^2(r) = \frac{F(r)}{r} + f. \tag{25}$$

We assume both f and v to be sufficiently smooth. We shall first obtain the conditions in terms of the initial data $f(r)$ and $\rho(r)$, which will then be translated in terms of the density and velocity functions. The center of the cloud is taken to be at rest in any spherically symmetric profile. We then have from (25) that

$$v(0) = f(0) = f'(0) = 0.$$

To analyse the structure of the sets of initial density and velocity that lead to naked singularity, we consider the cartesian product $D \times V$, where D is as in §4 and V is the set of all sufficiently differentiable (up to order four, as derivatives up to this order only will be involved) initial velocity functions v (with $v(0) = 0$, $v'(0) > 0$).

Simialr to D, we define a metric on V as

$$d_1(v, v_1) = \max_i \sup_{0 \le r \le M} |v^{(i)}(r) - v_1^{(i)}(r)|, \quad i = 0, 1, 2, 3, 4.$$

With the help of metrics d and d_1, we define a product topology on $D \times V$ in the usual manner. $D \times V$ also becomes a metric space under the metric

$$d_2((\rho, v), (\rho_1, v_1)) = d(\rho, \rho_1) + d_1(v, v_1).$$

We can assume [4] that $f''(0) \neq 0$ as well as $(1 + 3f''(0)/(2\rho(0)) \neq 0$. In terms of v these become $v'(0)^2 \neq \rho(0)/3$, and $v'(0) \neq 0$ respectively. We have

$$\lim_{r\to 0} p = \frac{3f''(0)}{F'''(0)} = \frac{3f''(0)}{2\rho(0)}. \quad \text{Therefore} \quad \lim_{r\to 0} \frac{1}{(1+p)^{1/2}} = \frac{1}{\left[1 + \frac{3f''(0)}{2\rho(0)}\right]^{1/2}}, \tag{26}$$

$$\lim_{r\to 0} G(-p) = G\left(-\frac{3f''(0)}{2\rho(0)}\right). \tag{27}$$

Therefore

$$\Theta_0 = \left[1 + \frac{3f''(0)}{2\rho(0)}\right]^{-1/2} \lim_{r\to 0} \frac{\left(1 + \frac{rf'}{f} - \frac{rF'}{F}\right)}{r^{3(\alpha-1)/2}} + \lim_{r\to 0} \frac{\left(\frac{rF'}{F} - \frac{3rf'}{2f}\right)}{r^{3(\alpha-1)/2}} G\left(-\frac{3f''(0)}{2\rho(0)}\right). \tag{28}$$

We use the notations

$$g_1(r) \equiv 1 + \frac{rf'}{f} - \frac{rF'}{F} \tag{29}$$

and

$$g_2(r) \equiv \frac{rF'}{F} - \frac{3rf'}{2f}. \tag{30}$$

It follows that

$$\lim_{r\to 0} g_1(r) = \lim_{r\to 0} g_2(r) = 0.$$

So, for $\alpha > 1$

$$\lim_{r\to 0} \frac{g_1(r)}{r^{3(\alpha-1)/2}} = \lim_{r\to 0} \frac{g_1'}{\frac{3(\alpha-1)}{2} r^{(3(\alpha-1)/2)-1}},$$

$$\lim_{r\to 0} \frac{g_2(r)}{r^{3(\alpha-1)/2}} = \lim_{r\to 0} \frac{g_2'}{\frac{3(\alpha-1)}{2} r^{(3(\alpha-1)/2)-1}}.$$

Case 1: $f'''(0) \neq 0$ or $\rho'(0) \neq 0$. Since

$$\lim_{r\to 0} g_1'(r) = \frac{f'''(0)}{3f''(0)} - \frac{3\rho'(0)}{4\rho(0)}, \quad \lim_{r\to 0} g_2'(r) = -\frac{f'''(0)}{2f''(0)} + \frac{3\rho'(0)}{4\rho(0)},$$

it follows that in case either

$$\frac{f'''(0)}{3f''(0)} - \frac{3\rho'(0)}{4\rho(0)} \neq 0, \tag{31}$$

or

$$-\frac{f'''(0)}{2f''(0)} + \frac{3\rho'(0)}{4\rho(0)} \neq 0, \tag{32}$$

we can choose $\alpha = 5/3$ to get

$$\Theta_0 = \left[1 + \frac{3f''(0)}{2\rho(0)}\right]^{-1/2} \left[\frac{f'''(0)}{3f''(0)} - \frac{3\rho'(0)}{4\rho(0)}\right] + G\left(-\frac{3f''(0)}{2\rho(0)}\right) \times \left[\frac{3\rho'(0)}{4\rho(0)} - \frac{f'''(0)}{2f''(0)}\right]. \tag{33}$$

The condition (31) or (32) is satisfied if and only if $(f'''(0) \neq 0$ or $\rho'(0) \neq 0)$. We thus have that, when, either $f'''(0) \neq 0$ or $\rho'(0) \neq 0$, and further when the expression on right hand side of equation (33) is nonzero, we can choose $\alpha = 5/3$ to get a non-zero and finite value for Θ_0. We shall now find conditions under which equation (13) has positive finite root. Since, for $\alpha = 5/3$

$$P_0 \equiv \lim_{r\to 0} pr^{\alpha - 1} = 0, \quad \lambda_0 \equiv \lim_{r\to 0} \frac{F}{r^\alpha} = 0,$$

equation (13) becomes

$$X_0 - \frac{3}{5}\left[X_0 + \frac{\Theta_0}{\sqrt{X_0}}\right] = 0, \quad \text{i.e} \quad X_0^{3/2} = \frac{3}{2}\Theta_0, \tag{34}$$

and this will have positive root if and only if $\Theta_0 > 0$ in which case the singularity will be naked. In terms of velocity and density this case becomes: If $\rho'(0) \neq 0$ or $v''(0) \neq 0$ then the singularity is naked if one of the two conditions is satisfied.

(1) $0 < v'(0)^2 < \rho(0)/3$ and

$$\rho'(0)\left[-\frac{3v'(0)^2}{4\rho(0)} + G\left(1 - \frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2}\left(\frac{3v'(0)^2}{4\rho(0)} + \frac{1}{8}\right)\right] < v'(0)v''(0)\left[\frac{3}{2}G\left(1 - \frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2} - 1\right]. \tag{35}$$

(2) $\rho(0)/3 < v'(0)^2$ and

$$\rho'(0)\left[-\frac{3v'(0)^2}{4\rho(0)}+G\left(1-\frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2}\left(\frac{3v'(0)^2}{4\rho(0)}+\frac{1}{8}\right)\right]$$
$$> v'(0)v''(0)\left[\frac{3}{2}G\left(1-\frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2}-1\right]. \tag{36}$$

We consider the inequality (35). Let us choose $v(0)$ and $\rho(0)$ such that

$$\left(\frac{3v'(0)^2}{\rho(0)}\right) \ll 1.$$

Denote $(3v'(0)^2/\rho(0))$ by x. Inequality (35), then becomes

$$\rho'(0)\left[-\frac{x}{4}+G(1-x)\sqrt{x}\left(\frac{x}{4}+\frac{1}{8}\right)\right] < v'(0)v''(0)\left[\frac{3}{2}G(1-x)\sqrt{x}-1\right].$$

Using expression for $G(x)$ and neglecting x, x^2 etc. with respect to $x^{1/2}$, we then have

$$\rho'(0)\left[\sqrt{x}\pi/16\right] < v'(0)v''(0)\left[\frac{3\pi}{4}\sqrt{x}-1\right]. \tag{37}$$

Under the assumption that $x \ll 1$ we can have $3\pi x^{1/2}/4 < 1$. The inequality (37) is then equivalent to

$$\rho'(0) < v'(0)v''(0)\frac{16}{\pi\sqrt{x}}\left[\frac{3\pi}{4}\sqrt{x}-1\right]$$

as well as to

$$v''(0) > \frac{\rho'(0)\pi\sqrt{x}}{16v'(0)}\frac{1}{\left[\frac{3\pi}{4}\sqrt{x}-1\right]}$$

and from these we can choose either $\rho'(0)$ or $v''(0)$ if the other is known.

It is thus clear that given initial density profile, we can choose initial velocity field and vice-versa so that the singularity is naked. Reversing the signs in inequalities (35), (36) one may get black holes and following the above argument, it follows that given initial density profile, we can choose initial velocity field and vice-versa so that the singularity is covered.

It is also clear from the earlier inequalities that even if we have initial density ρ such that $\rho'(0) = \rho''(0) = \rho'''(0)$, we can choose initial velocity v such that the singularity is naked or covered. This result is in contrast with the corresponding result for the marginally bound collapse.

Consider now, the set of all those (ρ, v) satisfying either of the conditions (35), (36). Using continuity of real valued maps $\rho \to \rho^{(i)}(0)$, and $v \to v^{(i)}(0), i = 0,1,2,3,4$ we can show that the set of (ρ, v) satisfying (35) is open and same holds for that satisfying (36). This can be shown by considering composition of a number of continuous maps defined above and the set under consideration will be an inverse image of an open interval under a composition of continuous maps. It is then clear that the naked singularity (as well as black hole, following similar argument) forming in this case of initial data is stable, but the set of initial data is non-generic.

Case 2: $f'''(0)=0, \rho'(0)=0, (f''''(0)\neq 0$ or $\rho''(0)\neq 0)$. We have to choose $\alpha > 5/3$ and therefore

$$\lim_{r\to 0}\frac{g_1(r)}{r^{3(\alpha-1)/2}} = \lim_{r\to 0}\frac{g_1''}{[3(\alpha-1)/2][3(\alpha-1)/2-1]r^{3(\alpha-1)/2-2}}$$

and

$$\lim_{r\to 0}\frac{g_2(r)}{r^{3(\alpha-1)/2}} = \lim_{r\to 0}\frac{g_2''}{[3(\alpha-1)/2][3(\alpha-1)/2-1]r^{3(\alpha-1)/2-2}}.$$

Since

$$\lim_{r\to 0} g_1''(r) = \frac{f''''(0)}{3f''(0)} - \frac{6\rho''(0)}{5\rho(0)}, \tag{38}$$

$$\lim_{r\to 0} g_2''(r) = \frac{f''''(0)}{2f''(0)} + \frac{6\rho''(0)}{5\rho(0)}, \tag{39}$$

we have either

$$\lim_{r\to 0} g_1''(r)\neq 0 \quad \text{or} \quad \lim_{r\to 0} g_2''(r)\neq 0$$

and therefore we can choose $\alpha = 7/3$ to get

$$\Theta_0 = \left[1+\frac{3f''(0)}{2\rho(0)}\right]^{-1/2}\left[\frac{f''''(0)}{6f''(0)} - \frac{3\rho''(0)}{5\rho(0)}\right] + G\left(-\frac{3f''(0)}{2\rho(0)}\right) \times \left[\frac{3\rho''(0)}{5\rho(0)} - \frac{f''''(0)}{4f''(0)}\right], \tag{40}$$

provided the r.h.s. does not vanish. So with $\alpha = 7/3$, we have $P_0 = 0, \lambda_0 = 0$, and equation (13) becomes

$$X_0 - \frac{3}{7}\left(X_0 + \frac{\Theta_0}{\sqrt{X_0}}\right) = 0 \quad \text{i.e.} \quad X_0^{3/2} = \frac{3\Theta_0}{4}. \tag{41}$$

This equation will have finite positive root if and only if $\Theta_0 > 0$, and this will mean that the singularity of collapse is naked. In terms of density and velocity this condition then becomes: If $\rho'(0) = v''(0) = 0, (\rho''(0)\neq 0$ or $v'''(0)\neq 0)$ then the singularity is naked if one of the two conditions is satisfied:

(1) $0 < v'(0)^2 < \rho(0)/3$ and

$$\rho''(0)\left[-\frac{3v'(0)^2}{5\rho(0)} + G\left(1-\frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2}\left(\frac{3v'(0)^2}{5\rho(0)}+\frac{1}{10}\right)\right] < v'(0)v'''(0)\left[G\left(1-\frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2} - (2/3)\right], \tag{42}$$

or

(2) $\rho(0)/3 < v'(0)^2$ and

$$\rho''(0)\left[-\frac{3v'(0)^2}{5\rho(0)} + G\left(1-\frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2}\left(\frac{3v'(0)^2}{5\rho(0)}+\frac{1}{10}\right)\right]$$
$$> v'(0)v'''(0)\left[G\left(1-\frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2} - (2/3)\right]. \tag{43}$$

Because of the condition $\rho'(0) = v''(0) = 0$, it is clear that, the set of initial data leading to naked singularity (or black hole by reversing the inequalities) is neither open (see §4, case 2 as in this case also one can define slight perturbations of density or the velocity so that the condition $\rho'(0) = v''(0) = 0$ is not satisfied), nor is it dense. Thus the naked singularity (or black hole) formed in this case is unstable and the initial data leading to it is non-generic. The instability arising here is also to be considered as an anomalous case, and as noted earlier is essentially due to the vanishing of the derivatives at $r = 0$.

Case 3: $f'''(0) = \rho'(0) = f''''(0) = \rho''(0) = 0, (f'''''(0) \neq 0$ or $\rho'''(0) \neq 0)$. Since

$$\lim_{r\to 0} g_1'''(r) = \frac{3f'''''(0)}{10f''(0)} - \frac{3\rho'''(0)}{2\rho(0)}, \tag{44}$$

$$\lim_{r\to 0} g_2'''(r) = -\frac{9f'''''(0)}{20f''(0)} + \frac{3\rho'''(0)}{2\rho(0)}, \tag{45}$$

we have

$$\lim_{r\to 0} g_1'''(r) \neq 0 \quad \text{or} \quad \lim_{r\to 0} g_2'''(r) \neq 0.$$

We can choose $\alpha = 3$ to get

$$\Theta_0 = \left[1+\frac{3f''(0)}{2\rho(0)}\right]^{-1/2}\left[\frac{f'''''(0)}{20f''(0)} - \frac{\rho'''(0)}{4\rho(0)}\right] + G\left(-\frac{3f''(0)}{2\rho(0)}\right)$$
$$\times\left[\frac{\rho'''(0)}{4\rho(0)} - \frac{3f'''''(0)}{40f''(0)}\right], \tag{46}$$

provided the r.h.s. is non-zero.

With $\alpha = 3$, we have $P_0 = 0$, $\lambda_0 = \rho(0)/3$ and equation (13) now becomes

$$X_0 - \left[1-\frac{\sqrt{\rho(0)}}{\sqrt{3X_0}}\right]\left[\frac{X_0+\frac{\Theta_0}{\sqrt{X_0}}}{3}\right] = 0$$

$$\text{i.e.} \quad 2X_0^2 + \frac{\sqrt{\rho(0)}}{\sqrt{3}}X_0^{3/2} - \Theta_0\sqrt{X_0} + \sqrt{\frac{\rho(0)}{3}}\Theta_0 = 0. \tag{47}$$

Substitute

$$x = \sqrt{\frac{3}{\rho(0)} X_0}$$

to get

$$2x^4 + x^3 + \xi x - \xi = 0, \tag{48}$$

where $\xi = -\Theta_0(3/\rho(0))^{3/2}$ and Θ_0 as given by (46).

The quartic equation (48) has positive root if and only if $\xi < \xi_1 = -25.9904$ i.e. if and only if

$$\left[1 + \frac{3f''(0)}{2\rho(0)}\right]^{-1/2} \left[\frac{f'''''(0)}{20f''(0)} - \frac{\rho'''(0)}{4\rho(0)}\right] + G\left(-\frac{3f''(0)}{2\rho(0)}\right) \times \left[\frac{\rho'''(0)}{4\rho(0)} - \frac{3f'''''(0)}{40f''(0)}\right] > (25.9904)\left(\frac{\rho(0)}{3}\right)^{3/2}. \tag{49}$$

In terms of density and velocity this becomes: If $(\rho'(0) = v''(0) = \rho''(0) = v'''(0) = 0)$, $(\rho'''(0) \neq 0$ or $v''''(0) \neq 0)$ then the singularity is naked if one of the two conditions is satisfied:

(1) $0 < v'(0)^2 < \rho(0)/3$ and

$$\rho'''(0)\left[-\frac{v'(0)^2}{4\rho(0)} + G\left(1 - \frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2}\left(\frac{v'(0)^2}{4\rho(0)} + \frac{1}{24}\right)\right] < v'(0)v''''(0)\left[\frac{3}{8}G\left(1 - \frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2} - (1/4)\right] + (25.9904)\frac{\rho(0)}{3}v'(0)\left(v'(0)^2 - \frac{\rho(0)}{3}\right) \tag{50}$$

or

(2) $\rho(0)/3 < v'(0)^2$ and

$$\rho'''(0)\left[-\frac{v'(0)^2}{4\rho(0)} + G\left(1 - \frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2}\left(\frac{v'(0)^2}{4\rho(0)} + \frac{1}{24}\right)\right] > v'(0)v''''(0)\left[\frac{3}{8}G\left(1 - \frac{3v'(0)^2}{\rho(0)}\right)\left(\frac{3v'(0)^2}{\rho(0)}\right)^{1/2} - (1/4)\right] + (25.9904)\frac{\rho(0)}{3}v'(0)\left(v'(0)^2 - \frac{\rho(0)}{3}\right). \tag{51}$$

From the argument in the previous case, it is clear that the naked singularity (or black hole by reversing the inequalities) formed in this case is unstable and the set of initial data is non-generic. The two examples studied in ([14], pp. 1231) and ([15], pp. 5363) belong to this case. In each case the density ρ is constant. However, in each case there is a freedom to choose constants M_0, e_0, e_1 (in first example, where $f(r) = -M_0 r^2(e_0 + e_1 r^3 + \gamma_0(r) r^3)$, and F_0, f_0, f_1 (in second example, where $f(r) = -f_0 r^2 + f_0 f_1 r^5$) such that the singularity is naked or otherwise. However, each example leads to unstable black hole or naked singularity as is inherent in this case due to vanishing of some of the derivatives at $r = 0$. As in the earlier case this is also an anomalous case.

Case 4: $f'''(0) = \rho'(0) = f''''(0) = \rho''(0) = f'''''(0) = \rho'''(0) = 0$. In this case we have to choose α greater than 3 which makes $\lambda_0 = \infty$ and eq. (13) cannot have finite positive root and the collapse ends into a black hole. This is also an anomalous case where black holes are unstable.

6. Concluding remarks

1. Stability results derived in [16] by using perturbation methods are not directly related to our stability results though there may be a distant relationship between the two, and this needs detailed investigation. For example, given a density distribution ρ_0 corresponding to certain dust metric g_0, let us consider ρ_1 in the neighborhood of ρ_0 with respect to a given norm. This ρ_1 on the initial surface is related to F_1 and R_1 by the relation $\rho_1 = F_1'/(R_1^2 R_1')$. Thus F_1 and R_1 are different from F_0 and R_0 involved in $\rho_0 = F_0'/(R_0^2 R_0')$ and hence will give rise to a metric g_1 different from g_0. Thus g_1 can be thought of as a perturbation of g_0. Hence stability with respect to initial data of a naked singularity (or a black hole) is also stability with respect to perturbation in a space-time metric in the above sense.

2. Definition of stability adopted here is the same as that adopted in our recent work [17]. We hope to apply same definition to prove stability results in a general form of matter following the results in [7]. Here initial data constitutes four functions and condition for collapse ending in a naked singularity or otherwise needs to be expressed explicitly in terms of these functions. Work is in progress in this direction.

References

[1] C J S Clarke and A Krolak, *J. Geom. Phys.* **2**, 127 (1985)
[2] R P A C Newman, *Class. Quantum. Gravit.* **3**, 527 (1986)
[3] P S Joshi, *Global Aspects in Gravitation and Cosmology* (Clarendon Press, Oxford, 1993)
[4] T P Singh and P S Joshi *Class. Quantum Gravit.* **13**, 559 (1996)
[5] I H Dwivedi and P S Joshi, *Commun. Math. Phys.* **166**, 117 (1994)
[6] F I Cooperstock, S Jhingan, P S Joshi and T P Singh, *Class. Quantum Gravit.* **14**, 2195 (1997)
[7] P S Joshi and I H Dwivedi, *Class. Quantum Gravit.* **16**, 41 (1999)
[8] R C Tolman, *Proc. Natl. Acad. Sci. U.S.A* **20**, 410 (1934)
[9] H Bondi, *Mon. Not. Astron. Soc.* **107**, 343 (1947)
[10] R Abraham and J Marsden, *Foundations of Mechanics*, 2nd edition (The Benjamin/Cummings Publishing Company, 1978)
[11] T P Singh, *Phys. Rev.* **D58**, 108502 (1998)
[12] S Jhingan and P S Joshi, *Ann. Israel Phys. Soc.* **13**, 357 (1998)
[13] S Jhingan, P S Joshi and T P Singh, *Class. Quantum Gravit.* **13**, 3057 (1996)
[14] I H Dwivedi and P S Joshi, *Class. Quantum Gravit.* **14**, 1223 (1997)

[15] P S Joshi and I H Dwivedi, *Phy. Rev.* **D47**, 5357 (1993)
[16] H Iguchi, K Nakao and T Harada, *Phy. Rev.* **D58**, 041502 (1998)
[17] R V Saraykar and S H Ghate, *Class. Quantum Gravit.* **16** (1999)