

Some extensions and applications
of Eisenstein Irreducibility
Criterion

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Eisenstein Irreducibility Criterion.(1850)

Let $F(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ be a polynomial with coefficient in the ring \mathbb{Z} of integers. Suppose that there exists a prime number p such that

- a_0 is not divisible by p ,
- a_i is divisible by p for $1 \leq i \leq n$,
- a_n is not divisible by p^2

$F(x)$ is irreducible over the field \mathbb{Q} of rational numbers.

Example: Consider the p th cyclotomic polynomial $x^{p-1} + x^{p-2} + \dots + x + 1 = \frac{x^p - 1}{x - 1}$. On changing x to $x + 1$ it becomes $\frac{(x + 1)^p - 1}{(x + 1) - 1} = x^{p-1} + \binom{p}{1} x^{p-2} + \dots + \binom{p}{p-1}$ and hence is irreducible over \mathbb{Q} .

This slick proof of the irreducibility for the p th cyclotomic polynomial was given by the **Eisenstein**, though its irreducibility was proved by **Gauss** in 1799.

In 1906, **Dumas** proved the following generalization of this criterion.

Dumas Criterion. *Let $F(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ be a polynomial with coefficients in \mathbb{Z} . Suppose there exists a prime p whose exact power p^{r_i} dividing a_i (where $r_i = \infty$ if $a_i = 0$), $0 \leq i \leq n$, satisfy*

- $r_0 = 0$,
- $(r_i/i) > (r_n/n)$ for $1 \leq i \leq n - 1$ and
- $\gcd(r_n, n)$ equals 1.

Then $F(x)$ is irreducible over \mathbb{Q} .

Example : $x^3 + 3x^2 + 9x + 9$ is irreducible over \mathbb{Q} .

Note that Eisenstein's criterion is a special case of Dumas Criterion with $r_n = 1$.

For a given prime number p , let v_p stand for the mapping $v_p : \mathbb{Q}^* \rightarrow \mathbb{Z}$ defined as follows. Write any non zero rational number $x = p^r \frac{a}{b}$, $p \nmid ab$. Set $v_p(x) = r$. Then

$$(i) \ v_p(xy) = v_p(x) + v_p(y)$$

$$(ii) \ v_p(x + y) \geq \min\{v_p(x), v_p(y)\}.$$

Set $v_p(0) = \infty$. v_p is called the p-adic valuation of \mathbb{Q} .

Dumas Criterion. *Let $F(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ be a polynomial with coefficients in \mathbb{Z} . Suppose there exists a prime number p such that $v_p(a_0) = 0$, $v_p(a_i)/i > v_p(a_n)/n$ for $1 \leq i \leq n - 1$ and $v_p(a_n)$ is coprime to n , then $F(x)$ is irreducible over \mathbb{Q} .*

In 1923, Dumas criterion was extended to polynomials over more general fields namely, fields with discrete valuations by Kürschák. Indeed it was the Hungarian Mathematician **JOSEPH KÜRSCHÁK** who formulated the formal definition of the notion of valuation of a field in 1912.

Definition :A real valuation v of a field K is a mapping $v : K^* \rightarrow \mathbb{R}$ satisfying

$$(i) \ v(xy) = v(x) + v(y)$$

$$(ii) \ v(x + y) \geq \min\{v(x), v(y)\}$$

$$(iii) \ v(0) = \infty.$$

$v(K^*)$ is called the value group of v . A real valuation is said to be discrete if $v(K^*)$ is isomorphic to \mathbb{Z} .

In 1931, Krull generalized the above notion of valuation.

By a Krull valuation of a field K we mean a mapping

$$v : K^* \rightarrow G$$

where G is a totally ordered (additively written) abelian group satisfying (i), (ii) and (iii). The pair (K, v) is called a valued field. The subring $\mathcal{R}_v = \{x \in K \mid v(x) \geq 0\}$ of K is called the valuation ring of v . It has a unique maximal ideal given by $\mathcal{M}_v = \{x \in K \mid v(x) > 0\}$. R_v/\mathcal{M}_v is called the residue field of v . For any ξ belonging to R_v $\bar{\xi}$ will stand for the canonical homomorphism from R_v onto R_v/\mathcal{M}_v .

Example. Let v_x denote the x -adic valuation of the field $\mathbb{Q}(x)$ of rational functions in x trivial on \mathbb{Q} and v_p denote the p -adic valuation of \mathbb{Q} . For any non-zero polynomial $f(x)$ belonging to $\mathbb{Q}(x)$, we shall denote by f^* the constant term of the polynomial $f(x)/x^{v_x(f(x))}$. Let v be the mapping from non-zero elements of $\mathbb{Q}(x)$ to $\mathbb{Z} \times \mathbb{Z}$ (lexicographically ordered) defined on $\mathbb{Q}[x]$ by

$$v(f(x)) = (v_x(f(x)), v_p(f^*)).$$

Then v gives a valuation on $\mathbb{Q}(x)$.

In 1997, Saha and S.K.K generalized the Eisenstein-Dumas-Kürschák criterion.

Theorem 1. (-, Saha) *let v be a Krull valuation of a field K with value group G and $F(x) = a_0x^s + a_1x^{s-1} + \dots + a_s$ be a polynomial over K . If*

- $v(a_0) = 0$,
 - $v(a_i)/i \geq v(a_s)/s$ for $1 \leq i \leq s$ and
 - *there does not exist any integer $d > 1$ dividing s such that $v(a_s)/d \in G$,*
- then $F(x)$ is irreducible over K .*

Definition. A polynomial which satisfies the hypothesis of Theorem 1 is called an Eisenstein-Dumas polynomial with respect to v .

Example: Let $F(X, Y) = g(Y)X^s + h(Y)$ be a polynomial over a field L in independent variables X, Y . If $g(Y), h(Y)$ have no common factors and $\deg g(Y) - \deg h(Y)$ is coprime to s , then $F(X, Y)$ is irreducible over L .

Verification: Regard $F(X, Y)/g(Y)$ as a polynomial in X with coefficients over the field $K = L(Y)$ with valuation on K defined by $v(a(Y)/b(Y)) = \deg b(Y) - \deg a(Y)$ and apply the criterion by Saha.

In 2001, S. Bhatia generalized Eisenstein's Irreducibility Criterion in a different direction.

Theorem 2. (–, S. Bhatia) *Let v be a valuation of a field K with value group the set of integers. Let $g(x) = x^m + a_1x^{m-1} + \dots + a_m$ be a polynomial with coefficients in K such that $v(a_i)/i > v(a_m)/m$ for $1 \leq i \leq m-1$. Let r denote $\gcd(v(a_m), m)$ and b be an element of K with $v(b) = v(a_m)/r$. Suppose that the polynomial $z^r + (a_m/b^r)$ in the indeterminate z is irreducible over the residue field of v . Then $g(x)$ is irreducible over K .*

Theorem 3. *Let $f(x)$ and $g(y)$ be non-constant polynomials with coefficients in a field k . Let c and c_0 denote respectively the leading coefficients of $f(x)$ and $g(y)$ and n, m their degrees. If $\gcd(m, n) = r$ and if $z^r - (c_0/c)$ is irreducible over k , then so is $f(x) - g(y)$.*

The result of Theorem 3 has its roots in a theorem of Ehrenfeucht. In 1956, **Ehrenfeucht** proved that a polynomial $f_1(x_1) + \dots + f_n(x_n)$ with complex coefficients is irreducible provided the degrees of $f_1(x_1), \dots, f_n(x_n)$ have greatest common divisor one.

In 1964, **Tverberg** extended this result by showing that when $n \geq 3$, then $f_1(x_1) + \dots + f_n(x_n)$ belonging to $K[x_1, \dots, x_n]$ is irreducible over any field K of characteristic zero in case the degree of each f_i is positive. Of course this result is false when characteristic of K is $p > 0$. Note that if a polynomial F can be written as $F = (g_1(x_1))^p + (g_2(x_2))^p + \dots + (g_n(x_n))^p + c[g_1(x_1) + g_2(x_2) + \dots + g_n(x_n)]$ where c is in K and each $g_i(x_i)$ is in $K[x_i]$, then it is reducible over K .

In 1966, Tverberg proved that the converse of the above simple fact holds in the particular case when $n = 3$ and K is an algebraically closed field of characteristic $p > 0$. In 1982, Schinzel extended Tverberg's result by showing that this converse holds for any $n \geq 3$. In 2004, Amrit Pal has given a proof of Schinzel's result which is shorter and entirely different from Schinzel's proof.

Question: *When is a translate $g(x+a)$ of a given polynomial $g(x)$ with coefficients in a valued field (K, v) an Eisenstein-Dumas polynomial with respect to v ?*

In 2009, we have characterized such polynomials using distinguished pairs.

Theorem 4 (-, Anuj Bishnoi). *Let v be a henselian Krull valuation of a field K . Let $g(x)$ belonging to $R_v[x]$ be a monic polynomial of degree e having a root θ . Then for an element a of K , $g(x+a)$ is an Eisenstein-Dumas polynomial with respect to v if and only if (θ, a) is a distinguished pair and $K(\theta)/K$ is a totally ramified extension of degree e .*

The following result which generalizes a result of M. Juras [12] proved in 2006 has been quickly deduced from the above theorem.

Theorem 5. *Let $g(x) = \sum_{i=0}^e a_i x^i$ be a monic polynomial with coefficients in a henselian valued field (K, v) . Suppose that the characteristic of the residue field of v does not divide e . If there exists an element b belonging to K such that $g(x + b)$ is an Eisenstein-Dumas polynomial with respect to v , then so is $g(x - \frac{a_{e-1}}{e})$.*

Classical Schönemann Irreducibility

Criterion. (1846) *If a polynomial $F(x)$ belonging to $\mathbb{Z}[x]$ has the form $F(x) = \phi(x)^s + pM(x)$ where p is a prime number,*

- *$\phi(x)$ belonging to $\mathbb{Z}[x]$ is a monic polynomial which is irreducible modulo p ,*

- *$\phi(x)$ is co-prime to $M(x)$ modulo p ,*
and

- *the degree of $M(x)$ is less than the degree of $F(x)$,*

then $F(x)$ is irreducible in $\mathbb{Q}[x]$.

Eisenstein's Criterion is easily seen to be a particular case of Schönemann Criterion by setting $\phi(x) = x$.

In 1997, Saha gave a generalization of Classical Schönemann Irreducibility Criterion using the theory of prolongations of a valuation defined on K to a simple transcendental extension of K which was initiated by MacLane and developed further by Popescu et al. In 2008, [Ron Brown](#) has given a different proof of Saha's result.

Recently, we have extended the Generalized Schönemann-Eisenstein Irreducibility Criterion.

Theorem 6. (-, R. Khassa) *Let v be a discrete valuation of K with value group \mathbb{Z} and π be an element of K with $v(\pi) = 1$.*

Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree m such that $\bar{f}(x)$ is irreducible over R_v/\mathcal{M}_v . Let $F(x)$ belonging to $R_v[x]$ be a monic polynomial having $f(x)$ -expansion $\sum_{i=0}^n A_i(x)f(x)^i$. Assume that there exists $s \leq n$ such that π does not divide the content of $A_s(x)$, π divides the content of each $A_i(x)$, $0 \leq i \leq s - 1$ and π^2 does not divide the content of $A_0(x)$. Then $F(x)$ has an irreducible factor of degree sm over the completion (\hat{K}, \hat{v}) of (K, v) which is a Schönemann polynomial with respect to \hat{v} and $f(x)$.

Theorem 7. *Let $(K, v), \pi$ be as above and $F(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ be a polynomial over R_v satisfying the following conditions for an index $s \leq n - 1$.*

- (i) $\pi | a_i$ for $0 \leq i \leq s - 1$, $\pi^2 \nmid a_0$, $\pi \nmid a_s$.*
 - (ii) The polynomial $x^{n-s} + \bar{a}_{n-1}x^{n-s-1} + \dots + \bar{a}_s$ is irreducible over the residue field of v .*
 - (iii) $\bar{d} \neq \bar{a}_s$ for any divisor d of a_0 in R_v .*
- Then $F(x)$ is irreducible over K .*



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Born: 16 April 1823 in Berlin, Germany
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What attracted me so strongly and exclusively to mathematics, apart from the actual content, was particularly the specific nature of the mental processes by which mathematical concepts are handled. This way of deducing and discovering new truths from old ones, and the extraordinary clarity and self-evidence of the theorems, the ingeniousness of the ideas ... had an irresistible fascination for me. Beginning from the individual theorems, I grew accustomed to delve more deeply into their relationships and to grasp whole theories as a single entity. That is how I conceived the idea of mathematical beauty ...

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THANK YOU