Communication Complexity

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Introduction

A framework for studying communication required for computation when the input is distributed among various parties.

Naive: The parties pool their inputs at one of the processors.

Clever: Based on what they want to compute, the parties send each other messages.

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- **Naive:** The parties pool their inputs at one of the processors.
- **Clever:** Based on what they want to compute, the parties send each other messages.

Formal setting

Alice

Receives a string $\mathbf{x} \in \mathcal{X}$

$\Leftarrow$
$\Rightarrow$

Bob

Receives a string $\mathbf{y} \in \mathcal{Y}$

- Alice and Bob are randomized agents.
- They exchange messages in order to compute a function $f(\mathbf{x}, \mathbf{y})$.
- We allow a small probability of error.
- Goal: minimize the total number of bits transmitted.

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Motivation . . .

- An abstract model to study the communicaiton required for computation.
- A tool for showing lower bounds in several computational models.
- The study often requires deep understanding of computation using tools from combinatorics, coding theory, algebra, analysis, etc.

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Example

Alice

Receives $\mathbf{x} \in \{0,1\}^n$

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Receives $\mathbf{y} \in \{0,1\}^n$

Goal

Determine if $\mathbf{x} = \mathbf{y}$

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Strategy I

Alice

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$\mathbf{y} \in \{0,1\}^n$

Naive strategy

Alice sends $\mathbf{x}$ to Bob.
Bob tells Alice if $\mathbf{x} = \mathbf{y}$.

Cost

Requires $n + 1$ bits of communication.

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Suppose Alice and Bob are provided $\mathbf{z}$ chosen randomly from $\{0,1\}^n$.

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Algebraic stragegy

Alice sends $\mathbf{x} \cdot \mathbf{z} \pmod 2$ to Bob.
Bob checks if $\mathbf{x} \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z} \pmod 2$.

Cost

Requires two bits of communication.

- If $\mathbf{x} = \mathbf{y}$, then $\Pr[\text{Bob says 'yes'}] = 1$.
- If $\mathbf{x} \neq \mathbf{y}$, then $\Pr[\text{Bob says 'yes'}] = \frac{1}{2}$.

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Strategy II

Assume Alice and Bob know a good error correcting code $\mathcal{E} : \{0,1\}^n \to \{0,1\}^{10n}$ with distance, say, $3n$.

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Receives a $\mathbf{y} \in \{0,1\}^n$

- Alice picks an index $i \in \{1, 2, \ldots, 10n\}$.
- Alice sends i and $\mathcal{E}(\mathbf{x})_i$ to Bob.
- Bob checks if $\mathcal{E}(\mathbf{x})_i = \mathcal{E}(\mathbf{y})_i$.

- Does not use shared randomness.
- Requires $1 + \lceil \lg 10n \rceil$ bits of communication.
- If $\mathbf{x} \neq \mathbf{y}$, then $\Pr[\text{Bob says 'yes'}] \leq 0.7$.

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Also called the appointment scheduling problem.

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