

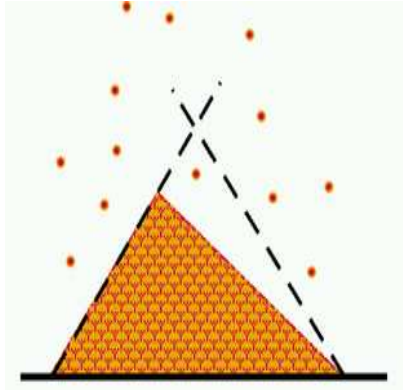
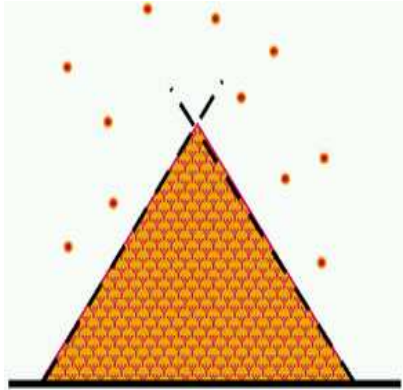
# Abelian Sandpile Model (ASM) and Infinite Volume Limit

Infinite volume limit for the stationary distribution of Abelian sandpile models.  
<http://www.isid.ac.in/~athreya/Research>

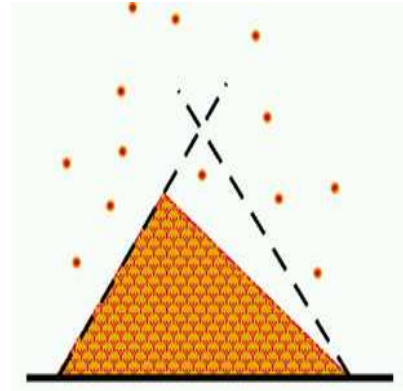
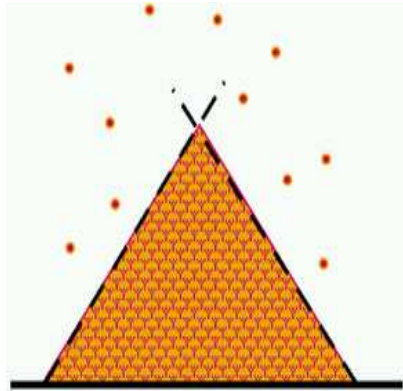
Joint work with Antal Jaraı.

*To Appear in Communications in Mathematical Physics*

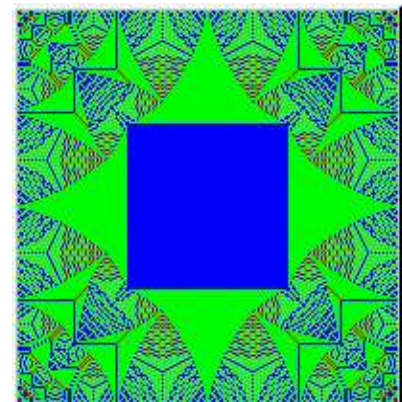
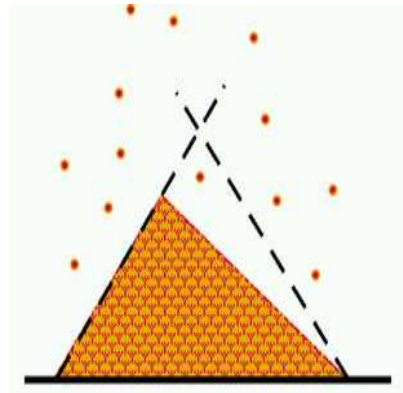
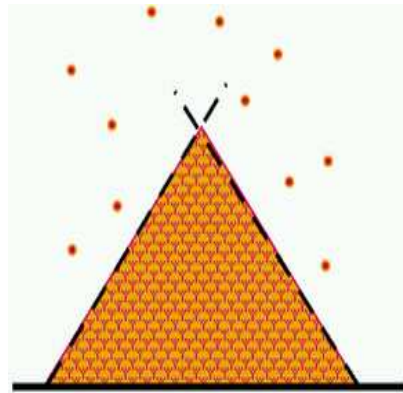
# Sandpile Images



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- SOC- was proposed by [BTW 87,88] as a mechanism which could explain occurrence of fractal structures in diverse natural phenomena.
- ASM is a simple model where SOC can be studied.
- Other SOC: Forest Fire models.

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- Repeat Step 1.

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1222

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1222

add a sand grain

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- (a) Add a particle at a randomly chosen site  $i \in \Lambda$ .
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- makes the model tractable and easy to analyze
- Due to the random choice of  $i$ , we have a Markov-chain with state space  $\Omega_\Lambda$ .
  - There is a unique stationary distribution  $\nu_\Lambda$ , which is uniform on the set of recurrent states of the Markov chain [Dhar 90].

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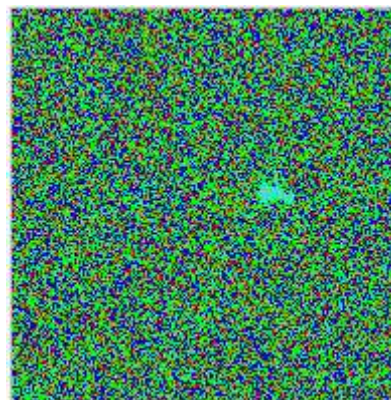
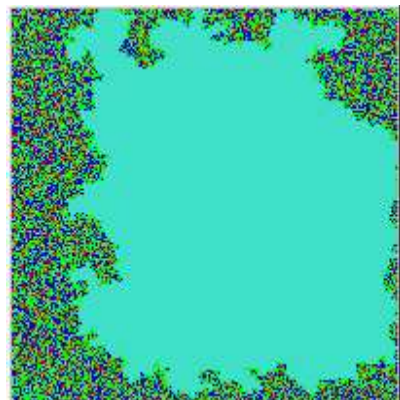
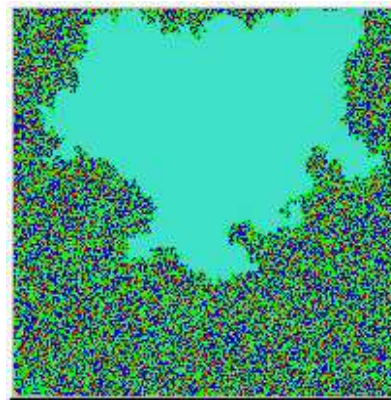
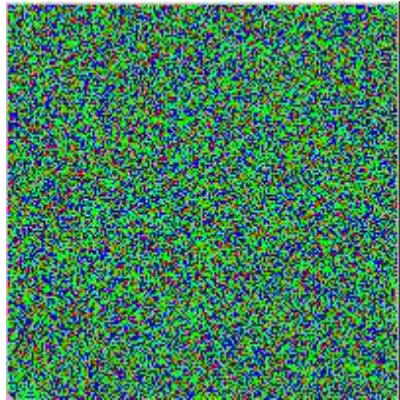
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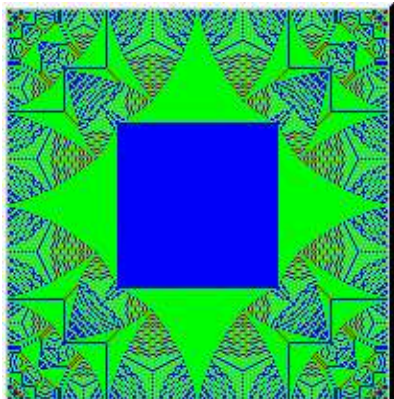
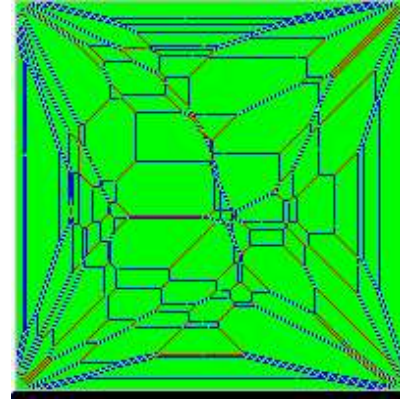
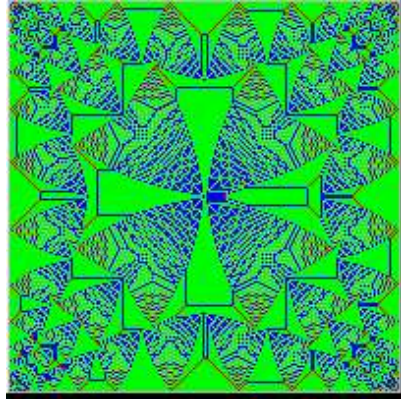
It is often assumed that these quantities have distributions with a power law tail in the limit  $\Lambda \nearrow \mathbb{Z}^d$ .

# ASM- Range of Avalanche

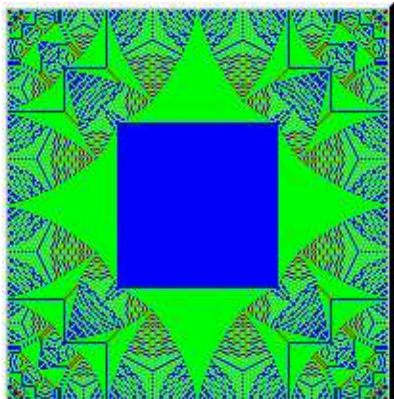
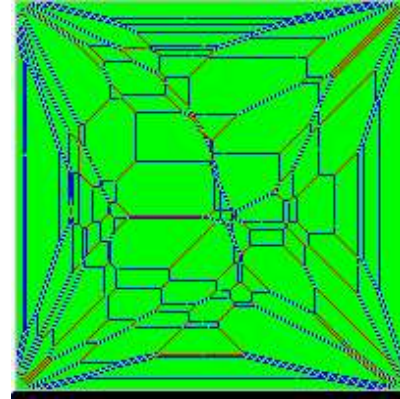
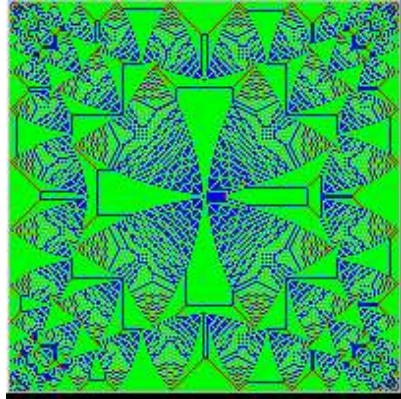


Source: <http://thy.phy.bnl.gov/www/xtoys.html>

# ASM- Recurrent and Identity State



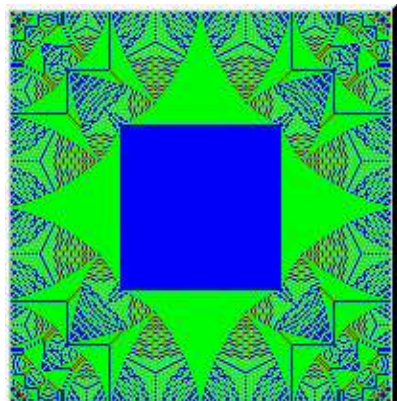
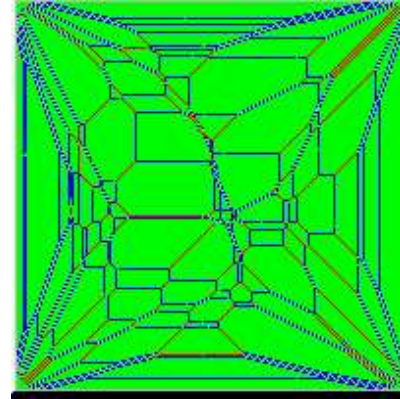
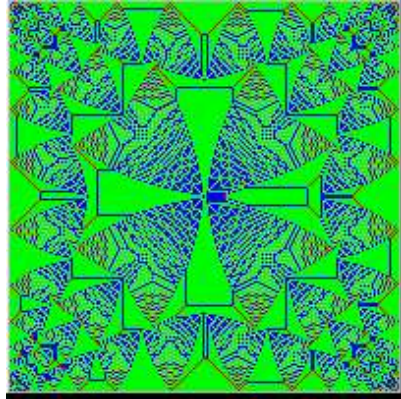
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- To the best of our knowledge, there is no rigorous proof of power law behavior, either in  $d = 2$  or higher.
  - As a step in analyzing the above distributions, we consider the limit  $\Lambda \nearrow \mathbb{Z}^d$ , and define avalanche characteristics in the infinite volume.

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- It is possible to show that  $\nu(S = s) > 0$
- From computations in [MD 91]: Under  $\nu$ , at least the random field  $I[z_x = 1]$  has power law correlations.

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**Theorem 1.** *Let  $d \geq 2$ . The measures  $\nu_n$  weakly converge to a translation invariant measure  $\nu$  on  $\Omega$ . For any cylinder event  $E$  and any  $v \in \mathbb{Z}^d$  we have*

$$\nu(E) = \lim_{n \rightarrow \infty} \nu_n(E) = \lim_{n \rightarrow \infty} \nu_n(\tau_v E)$$

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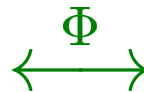
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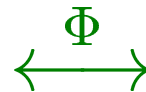


Wired  
Spanning  
Trees

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- We give expressions for  $\nu(E)$  in terms of the USF on  $\mathbb{Z}^d$ .

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Thank you for your attention