Stacked spheres and lower bound theorem

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- k points $v_1,\dots,v_k$ in $\mathbb{R}^N$ are called affinely independent if they do not lie on a $(k-2)$-dimensional affine subspace of $\mathbb{R}^N$.
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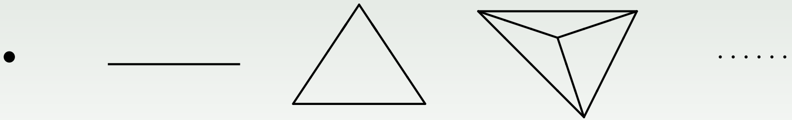

0-simplex (vertex) 1-simplex (edge) 2-simplex (triangle) 3-simplex (tetrahedron)

Definitions

A simplex α is called a **face** of a simplex β if each vertex of α is a vertex of β. (An i-dimensional face is also called an i-face.)

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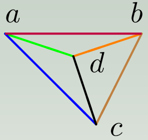
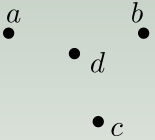
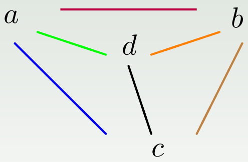


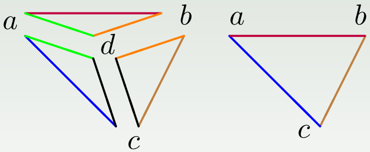
3-simplex β



0-faces (vertices) of β



1-faces (edges) of β



2-faces of β

Definitions

A collection X of simplices in some Euclidean space $\mathbb{R}^N$ is called a **triangulated d-manifold** if it satisfies the following:

(i) α is in X implies all the faces of α are in X.

(ii) If α and β are in X then either $\alpha \cap \beta$ is an empty set or $\alpha \cap \beta$ is a common face of α and β.

(iii) The space $|X| := \bigcup_{\beta \in X} \beta$ (union of all the simplices in X) is homeomorphic to a d-dimensional topological manifold M.

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If a collection X of simplices satisfies (i) and (ii) above then X is called a **simplicial complex**.

- A set $X \subseteq \mathbb{R}^m$ is called homeomorphic to a set $Y \subseteq \mathbb{R}^n$ if the exists an one-to-one correspondence $f\colon X \to Y$ s.t. both f & f^{-1} are continuous.

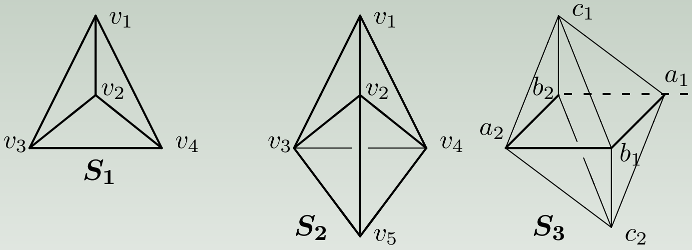
Examples

Examples of three triangulated 2-manifolds:

$$\begin{aligned} S_1 &= \{v_i, \langle v_i v_j\rangle, \langle v_i v_j v_k\rangle : 1 \leq i < j < k \leq 4\}, \\ S_2 &= \{v_i, \langle v_1 v_j\rangle, \langle v_j v_5\rangle, \langle v_1 v_k v_l\rangle, \langle v_k v_l v_5\rangle : 1 \leq i \leq 5, \\ &\qquad 2 \leq j \leq 4, 2 \leq k < l \leq 4\}, \\ S_3 &= \{a_i, b_j, c_k, \langle a_i b_j\rangle, \langle a_i c_k\rangle, \langle b_j c_k\rangle, \langle a_i b_j c_k\rangle : 1 \leq i, j, k \leq 2\}, \end{aligned}$$

where $v_1 = (1,1,1)$, $v_2 = (1,-1,-1)$, $v_3 = (-1,1,-1)$, $v_4 = (-1,-1,1)$, $v_5 = (-5/3,-5/3,-5/3)$, $a_1 = (1,0,0)$, $a_2 = (-1,0,0)$, $b_1 = (0,1,0)$, $b_2 = (0,-1,0)$, $c_1 = (0,0,1)$ and $c_2 = (0,0,-1)$ are points in $\mathbb{R}^3$.

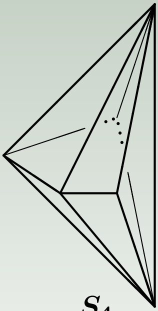
Examples



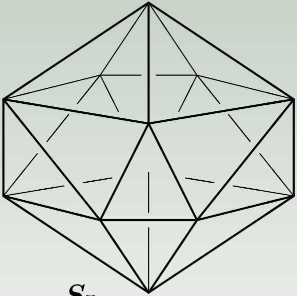
Clearly, S_i is a triangulation the 2-sphere S^2 and hence is a triangulated 2-manifold for $1 \leq i \leq 3$.
Observe that S_1 is the boundary complex of the tetrahedron and S_3 is the boundary complex of the octahedron.

Examples

Examples of two more triangulations of the 2-sphere S^2.



$\boldsymbol{S_4}$



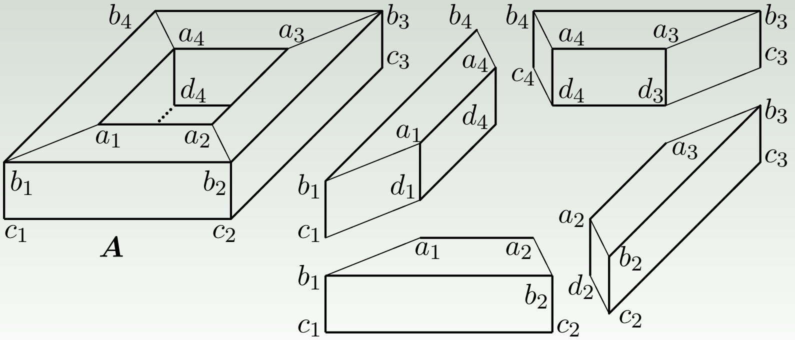
$\boldsymbol{S_5}$

The vertex set of S_4 is $\{(1,0,\pm 2), (\cos(\frac{2k\pi}{5}), \sin(\frac{2k\pi}{5}), 0) : 1 \leq k \leq 4\}$.
The vertex set of S_5 is $\{\pm(0,\tau,\pm 1), \pm(\pm 1, 0, \tau), \pm(\tau, \pm 1, 0)\}$, where $\tau = \frac{\sqrt{5}+1}{2}$. Note that S_5 is the boundary complex of the icosahedron.

Examples

Example of a triangulation of the torus. Let

$$\begin{aligned} A &= [([1,9]\times[0,8])\setminus((3,7)\times(2,6))]\times\{0,8\} \\ &\cup[([1,9]\times\{0,8\})\cup(\{1,9\}\times[0,8])]\times[0,8] \\ &\cup[([3,7]\times\{2,6\})\cup(\{3,7\}\times[2,6])]\times[0,8]\subseteq\mathbb{R}^3. \end{aligned}$$



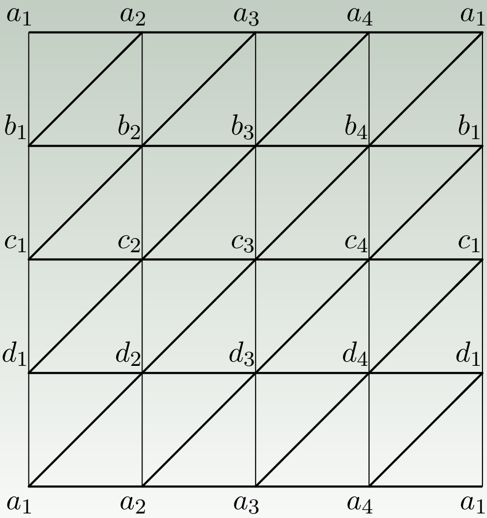
Examples

Then A is homeomorphic to the torus. Here $a_1 = (3, 2, 8)$, $a_2 = (7, 2, 8)$, $a_3 = (7, 6, 8)$, $a_4 = (3, 6, 8)$, $b_1 = (1, 0, 8)$, $b_2 = (9, 0, 8)$, $b_3 = (9, 8, 8)$, $b_4 = (1, 8, 8)$, $c_1 = (1, 0, 0)$, $c_2 = (9, 0, 0)$, $c_3 = (9, 8, 0)$, $c_4 = (1, 9, 0)$, $d_1 = (3, 2, 0)$, $d_2 = (7, 2, 0)$, $d_3 = (7, 6, 0)$, $d_4 = (3, 6, 0)$.

Now, consider the simplicial complex T whose vertices are $a_1, \dots, a_4$, $b_1, \dots, b_4$, $c_1, \dots, c_4$, $d_1, \dots, d_4$ and consists of 32 triangles given in the picture below. Check that T has 48 edges. Clearly, $|T| = A$. Therefore, T is a triangulation of the torus and hence is a triangulated 2-manifold.

Examples

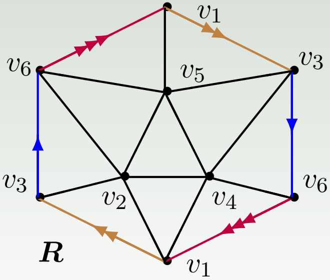
Triangles in *T*



Examples

Example of a triangulation of the real projective plane $\mathbb{R}P^2$. Consider the six points $v_1 = (1,0,0,0,0)$, $v_2 = (0,1,0,0,0)$, $v_3 = (0,0,1,0,0)$, $v_4 = (0,0,0,1,0)$, $v_5 = (0,0,0,0,1)$, $v_6 = (0,0,0,0,0)$ in $\mathbb{R}^5$. Let

$$R = \{v_1, \ldots, v_6, \langle v_1v_2\rangle, \ldots, \langle v_5v_6\rangle, \langle v_1v_2v_3\rangle, \langle v_1v_2v_4\rangle, \langle v_1v_3v_5\rangle, \langle v_1v_4v_6\rangle, \langle v_1v_5v_6\rangle, \langle v_2v_3v_6\rangle, \langle v_2v_4v_5\rangle, \langle v_2v_5v_6\rangle, \langle v_3v_4v_5\rangle, \langle v_3v_4v_6\rangle\}.$$



Examples

Then R is a triangulation of the real projective plane and hence is a triangulated 2-manifold. We have drown above a model of R on the Euclidean plane (same simplex appeared in two places means one has to identify them to get the actual simplicial complex).

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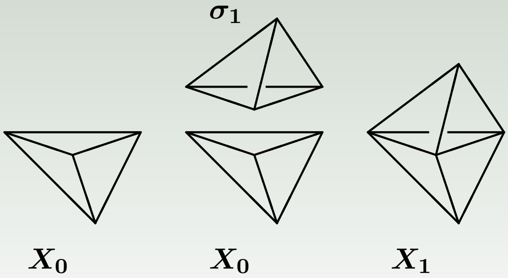
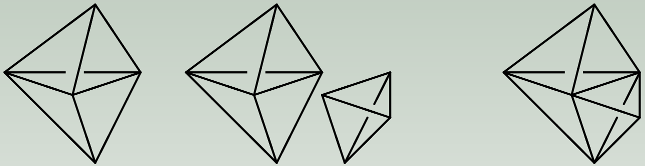
We have given here examples of few triangulated 2-manifolds. For several interesting higher dimensional triangulated manifolds see the articles given in the references.

Examples

We now construct a class of simplicial complexes (**stacked balls**) which triangulate the solid balls. A c-dimensional such ball is obtained from a c-simplex by adding more c-simplices successively.



Definitions



X_1 X_1 σ_2 X_2

Definition

A simplicial complex X is said to be a **stacked c-ball** if there exists a sequence of simplicial complexes $X_0, \ldots, X_m$ such that X_0 is a c-simplex, $X_m = X$ and for $1 \leq i \leq m$, X_i is obtained from X_{i-1} by adding a c-simplex σ_i to X_{i-1} such that σ_i intersect X_{i-1} on its boundary and $X_{i-1} \cap \sigma_i$ is a $(c-1)$-face of σ_i.

Definitions

If X is a stacked $(d+1)$-ball then $|X| := \bigcup_{\beta \in X} \beta$ is homeomorphic to the $(d+1)$-dimensional ball
$B^{d+1} := \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1^2 + \cdots + x_{d+1}^2 \leq 1\}$.

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Also, $|\partial X| := \bigcup_{\beta \in \partial X} \beta$ is homeomorphic to the d-dimensional sphere
$S^d := \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1^2 + \cdots + x_{d+1}^2 = 1\}$.

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The boundary $\boldsymbol{\partial X}$ of a triangulated $(d+1)$-ball $\boldsymbol{X}$ is the d-dimensional simplicial complex whose d-simplices are those α where α is a d-face of exactly one $(d+1)$-simplex in X.

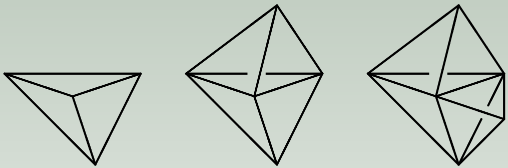
Definition

A triangulated d-manifold Y is said to be a **stacked d-sphere** if there exists a stacked $(d+1)$-ball X such that Y is isomorphic to ∂X.

Definitions

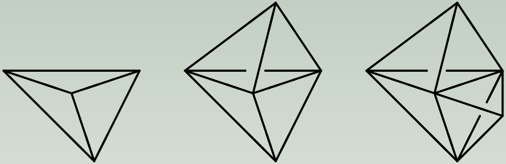
Two simplicial complexes X and Y are called isomorphic (denoted by $X \cong Y$) if there exists an one-to-one correspondence $f\colon$ vertex-set of $X \to$ vertex-set of Y such that $\langle v_1 v_2 \dots v_k \rangle$ is a simplex in X if and only if $\langle f(v_1) f(v_2) \dots f(v_k) \rangle$ is a simplex in Y.

Definitions & Examples



X_0 X_1 X_2

Definitions & Examples



$\boldsymbol{X_0}$ $\boldsymbol{X_1}$ $\boldsymbol{X_2}$

Let $f_i(X)$ denote the number of i-faces in a complex X. Then

	∂X_0	∂X_1	∂X_2	$\ldots$	∂X_m
f_0	4	$4+1$	$4+2$		$4+m$
f_1	$\binom{4}{2}=6$	$6+3$	$6+3+3$		$6+3m$
f_2	$\binom{4}{3}=4$	$4+(3-1)$	$4+2(3-1)$		$4+m(3-1)$

Definitions

In the general case, we have the following :
Let B be a stacked $(d+1)$-ball with n vertices. Let $X = \partial B$. Then B is obtained from a $(d+1)$-simplex by adding successively $n-d-2$ new $(d+1)$-simplices. Therefore, if $f_i(X)$ denotes the number of i-simplices in X (i.e., on the boundary of B) then

$$
\begin{aligned}
f_0(X) &= n \\
f_1(X) &= \binom{d+2}{2} + (n-d-2)(d+1), \\
f_j(X) &= \binom{d+2}{j+1} + (n-d-2)\binom{d+1}{j} \text{ for } 1 \leq j < d, \\
f_d(X) &= (d+2) + (n-d-2)d.
\end{aligned}
$$

Theorem

Theorem (Lower Bound Theorem)

Let X be any triangulated d-manifold with n vertices. Then

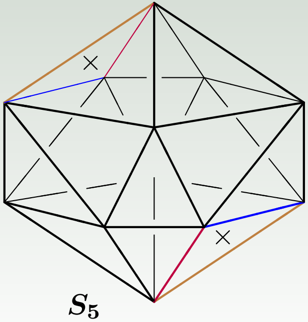
$$f_j(X) \geq \begin{cases} \binom{d+2}{j+1} + (n-d-2)\binom{d+1}{j}, & \text{if} \quad 1 \leq j < d, \\ (d+2) + (n-d-2)d, & \text{if} \quad j = d. \end{cases}$$

Further, for $d \geq 3$, equality holds here for some j if and only if X is a stacked sphere.

Walkup (1970), Barnette (1973), Klee (1975) proved various special cases of the LBT. In 1987, Kalai proved LBT for triangulated manifolds. Using Kalai's result, Tay (1995) proved LBT for a bigger class of simplicial complexes (namely, normal pseudomanifolds). In 2008, we (Bagchi & Datta) have presented a self-contained combinatorial proof of LBT for normal pseudomanifolds.

Definitions

Stacked spheres appeared initially as the boundary of stacked polytopes. They play important role in the theory of minimal triangulations of manifolds. For most of known triangulated manifolds with minimal number of vertices, the vertex-links are stacked spheres. Some of them are obtained from stacked spheres by elementary handle additions.



Minimal Triangulations

It is known (from the works of Walkup and Kühnel) that if X is a triangulated 4-manifold with Euler characteristic χ then $f_0(X)(f_0(X) - 11) \geq -15\chi$, with equality only for manifolds which can be obtained from stacked 4-spheres by elementary handle additions. There are such manifolds with 6 and 11 vertices. Next comes 15-vertex case. Is there a triangulated 4-manifold with 15 vertices and $\chi = -4$? We answered this questions affirmatively.

Theorem (Bagchi & Datta, 2011)

There exists a non-orientable 15*-vertex triangulated* 4*-manifold whose Euler characteristic is* -4.

Recently, my student Nitin Singh has constructed (with the help of computer) one such orientable 15-vertex triangulated 4-manifold.

Minimal Triangulations

It is known (from the works of Brehm and Kühnel) that if X is a non-simply connected triangulated d-manifold and $d \geq 3$ then it must contain at least $2d+3$ vertices. Kühnel also constructed a non-simply connected $(2d+3)$-vertex triangulated d-manifold for each $d \geq 2$. All these are obtained from stacked d-spheres by a single elementary handle addition.

Theorem (Bagchi & Datta, 2008)

There exists a unique non-simply connected d-manifold with $2d+3$ *vertices.*

We proved that to obtain such a d-manifold, there is a unique stacked d-sphere with $3d+4$ vertices and it allows a unique such elementary handle addition for $d \geq 3$.

Definitions

If $B = \sigma_0 \cup \sigma_1 \cup \cdots \cup \sigma_m$ is a stacked $(d+1)$-ball then we have:

1. All the $(d-1)$-simplex in B are in the boundary ∂B.
2. For each $i \geq 1$, $(\sigma_0 \cup \cdots \cup \sigma_{i-1}) \cap \sigma_i$ is one d-face of σ_i.

(In fact, these two conditions are equivalent.) Thus, we can define:

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(In fact, these two conditions are equivalent.) Thus, we can define:

Definition

For $0 \leq k \leq d+1$, a triangulated $(d+1)$-ball B is said to be ***k*-stacked** if all the $(d-k)$-simplices of B lie in its boundary ∂B.

Definition

For $0 \leq k \leq d+1$, a triangulated $(d+1)$-ball is said to be ***k-shelled*** if it is obtained from a $(d+1)$-simplex σ_0 by successively adding $(d+1)$-simplices $\sigma_1, \ldots, \sigma_m$ such that $(\sigma_0 \cup \cdots \cup \sigma_{i-1}) \cap \sigma_i$ is at most k d-faces of σ_i for $1 \leq i \leq m$.

Definitions

Definition

For $0 \leq k \leq d+1$, a triangulated d-manifold X is said to be a **k-stacked d-sphere** if X is isomorphic to the boundary ∂B of a k-stacked $(d+1)$-ball.

Definition

For $0 \leq k \leq d+1$, a triangulated d-manifold X is said to be a **k-stellated d-sphere** if X is isomorphic to the boundary ∂B of a k-shelled $(d+1)$-ball.

Recent Results

Recently, we have proved the following.

Theorem (Bagchi & Datta, 2011)

For each $k \geq 2$, there are k-stacked spheres which are not k-stellated.

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Theorem (Bagchi & Datta, 2011)

All k-stellated spheres are k-stacked.

We are now working on the class of triangulated manifolds whose vertex-links are k-stacked/k-stellated.

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End

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Thank You