Symmetry of solutions of differential equations

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Outline of the Talk

- Introduction to symmetry questions
- Basic framework for symmetry results
- Symmetry of solutions in a ball
- Solutions in the whole space
- More general framework for symmetry results
- conclusions

Nature Prefers symmetry

- Symmetry about a line, a plane, rotational symmetry, ...
- crystals, plants, flowers, insects.......
- yet, there are symmetry break ups !
- When is a profile symmetric ?
- If a physical phenomenon is modelled by a differential equation, when is the solution symmetric?
- Can we understand the threshold for symmetry ?

Basic framework : Laplacian

- Laplacian operator :

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

- Arises in many models :
  mechanics , electrostatics, ...

- Laplacian preserves rotational symmetry.

- Are the solutions radially symmetric in the whole space ?

- In a ball?

- For which nonlinearity?

- For which boundary conditions?

Laplacian in the unit circle

- Eigenfunction ϕ and eigenvalue λ :

$$-\Delta u = \lambda u \quad \text{in} \quad B$$

with boundary condition $u = 0$.

- First eigenfunction is positive and radially symmetric :

$$\phi_1(x) = J(r), \quad r = |x|$$

- Second eigenfunction is not !

$$\phi_2(x) = J(r)\ sin(\theta)$$

- Second eigenfunction changes sign.

- Are positive solutions radially symmetric?

Nonlinear equation

- Positive solution of

$$-\Delta u = f(u) \quad \text{in} \quad B$$

  with boundary condition $u = 0$.

- Arises as steady state equations for viscous incompressible flow;

- Solid cylindrical bar subjected to torsion ;

- Yang-Milles theory....

- Do positive solutions of this equation inherit the symmetry of the domain?

Radial symmetry

- Positive solution of

$$-\Delta u = f(u) \quad \text{in} \quad B$$

  with boundary condition $u = 0$.

- Gidas-Ni-Nirenberg :

  If f is differentiable, then u is radially symmetric.

- Proof : Use moving plane method and maximum principles in different forms.

- If $f(u) = u^\alpha$, for $0 < \alpha < 1$, then nonradial solutions can appear.

Ground state solution

- Positive solution of

$$-\Delta u = f(u) \quad \text{in} \quad I\!R^N$$

  vanishing as $|x|$ goes to infinity.

- f differentiable is not enough !

- Need additional conditions ;

- Difficulty : To start moving the plane from infinity.

- Gidas-Ni-Nirenberg :

  If f is differentiable and decreasing near 0, then u is radially symmetric.

- If f is differentiable and
  f is like u^α near 0, then
  fast decaying solutions are radially symmetric.

- There are slow decay, nonradial solutions.

More General framework

- More general operators : p- Laplacian :

$$-\Delta_p u = - \text{ div } (|Du|^{p-2} Du) = f(u)$$

  for $p > 1$, in a ball or in the whole space.

- Arises in the model for non-Newtonian fluid flow.

- Similar to Laplacian : maximum principle is available.

- Difficulties :
  Equation is nonlinear ;
  Coefficient $|Du|^{p-2}$ can disappear or become infinite ;
  Such difficult points are unknown ;
  Strong maximum principle is not available in full generality ;
  Solutions may not be twice differentiable.......

- Do symmetry results hold for this equation also?

Ground state solution for p - Laplacian

- Positive solutions of

  $$-\Delta_p u = -\text{ div } (|Du|^{p-2} Du) = f(u) \quad \text{in} \quad I\!R^N$$

  for 2 $p > 1$, vanishing as $|x|$ goes to infinity.

- Damascelli, Pacella, Mythily Ramaswamy :

  This solution in $I\!R^N$ is radially symmetric,

- if f is differentiable and decreasing near 0.

- If f is differentiable and
  f is like u^α near 0, then
  fast decaying solutions are radially symmetric.

Conclusions

For symmetry questions, we can consider

- Laplacian and more general elliptic operators satisfying maximum principle ;
- Dirichlet boundary conditions but not Neumann condition ;
- Bounded domains with symmetry or the whole space ;
- Even singular solutions ;
  solutions which blow up near the boundary or at some isolated point.
- Other geometries different from Euclidean geometry.

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