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Multipliers of weighted semigroups and associated Beurling Banach algebras

S J BHATT, P A DABHI and H V DEDANIA

Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120, India
E-mail: subhashbhaib@yahoo.co.in; lightatinfinite@gmail.com; hvdedania@yahoo.com

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Abstract. Given a weighted discrete abelian semigroup (S, ω), the semigroup $M_\omega(S)$ of ω-bounded multipliers as well as the Rees quotient $M_\omega(S)/S$ together with their respective weights $\tilde{\omega}$ and $\tilde{\omega}_q$ induced by ω are studied; for a large class of weights ω, the quotient $\ell^1(M_\omega(S), \tilde{\omega})/\ell^1(S, \omega)$ is realized as a Beurling algebra on the quotient semigroup $M_\omega(S)/S$; the Gel'fand spaces of these algebras are determined; and Banach algebra properties like semisimplicity, uniqueness of uniform norm and regularity of associated Beurling algebras on these semigroups are investigated. The involutive analogues of these are also considered. The results are exhibited in the context of several examples.

Keywords. Weighted semigroup; multipliers on a semigroup; generalized semicharacters; Beurling algebras; unique uniform norm property.

1. Introduction

The present paper contributes to harmonic analysis on semigroups with weights. Throughout, let S be a non-unital, faithful, abelian semigroup. A map $T : S \to S$ is a *multiplier* on S if

$$T(st) = sT(t) = T(s)t, \qquad s, t \in S.$$

The class of multipliers on S is denoted by $M(S)$. The set $M(S)$ is a unital, abelian semigroup under the operation composition. For $s \in S$, define $L_s : S \to S$ as $L_s(t) = st,\ t \in S$. Then $L_s \in M(S)$, and $L_s L_t = L_{st},\ s, t \in S$. A semigroup S is *faithful* if for $s, t \in S$, $su = tu$ for all $u \in S$ implies $s = t$. Thus S is identified with an ideal of $M(S)$ via $s \mapsto L_s$.

A *weight* on a semigroup S is a map $\omega : S \to (0, \infty)$ satisfying $\omega(st) \leq \omega(s)\omega(t)$, $s, t \in S$. A semigroup S with a weight ω is a *weighted semigroup* (S, ω). Then the number $\omega(s)$, for $s \in S$, presumably represents frequency or size of s in S. We regard ω on S as an analogue of a norm on a normed algebra; and the analogy is fruitfully pursued looking to (S, ω) as an intrinsic object. We call a multiplier $T \in M(S)$ *ω-bounded* if there exists $K > 0$ such that $\omega(Ts) \leq K\omega(s)$ for all $s \in S$. The set of ω-bounded multipliers on S will be denoted by $M_\omega(S)$. We note that $M_\omega(S)$ is a subsemigroup of $M(S)$; and S is imbedded in $M_\omega(S)$ via $s \mapsto L_s$. A $*$-*semigroup* is a semigroup with involution.

Let ω be a symmetric weight (i.e. $\omega(s^*) = \omega(s),\ s \in S$) on S. For $T \in M_\omega(S)$, define $T^* : S \to S$ as

$$T^*(s) = (Ts^*)^*, \qquad s \in S.$$

Then $M_\omega(S)$ is a $*$-semigroup and $\{L_s : s \in S\}$ is a $*$-ideal in $M_\omega(S)$.

We recall Rees quotient of S by an ideal I. The relation $\sim$ in S, defined by $s \sim t$ if either $s = t$ or both s and t are in I, is an equivalence relation in S. The equivalence classes under $\sim$ are the singleton sets $\{s\}$ with $s \in S \setminus I$ and the set I. Since I is an ideal of S, the relation $\sim$ is a congruence on S. The quotient semigroup S/I is the *Rees factor semigroup* of S modulo I [17].

Though the multipliers $M(S)$ ought to have been studied, we failed to find a systematic exposition. The ω-bounded multipliers have not been investigated at all. In the present paper, we investigate some basic aspects of $M_\omega(S)$. Notice that if S is unital, $S = M(S)$; and if $M_\omega(S) = S$, then S is unital. It is shown that if S admits a finite set of relative units, then $M_\omega(S) = M(S)$ for all ω; and that there exists (S, ω) such that $M_\omega(S) \neq M(S)$. It is also shown that S is separating if and only of $M_\omega(S)$ is separating; S is an inverse semigroup if and only if $M_\omega(S)$ is an inverse semigroup; and if S is involutive with ω symmetric, then S is $*$-separating if and only if $M_\omega(S)$ is $*$-separating. There exists S for which both S and $M(S)$ are separating but the quotient $M(S)/S$ fails to be separating. A weight ω on S induces a natural weight $\tilde{\omega}$ on $M_\omega(S)$ defined as

$$\tilde{\omega}(T) := \sup\left\{\frac{\omega(Ts)}{\omega(s)} : s \in S\right\} \quad (T \in M_\omega(S)),$$

resulting in weighted semigroups $(M_\omega(S), \tilde{\omega})$ as well as the quotient semigroup $M_\omega(S)/S$ with induced weight $\tilde{\omega}_q$. For $s \in S$, $\tilde{\omega}(L_s) \leq \omega(s)$; and in general, the equality does not hold. However, if ω is a uniform weight (i.e., $\omega(s^2) = \omega(s)^2,\ s \in S$) or a C^*-weight (i.e., $\omega(s^*s) = \omega(s)^2,\ s \in S$), then $\tilde{\omega}(L_s) = \omega(s),\ s \in S$; and $\tilde{\omega}$ is respectively a uniform weight or a C^*-weight. Also ω is semisimple if and only if $\tilde{\omega}$ is semisimple; and if $\tilde{\omega}$ is a Beurling–Domar weight, then so is ω, though the converse does not appear to hold. The converse is shown to hold under a stringent condition. A Beurling–Domar weight is a GRS (Gel'fand–Raikov–Šilov) weight; and there are GRS weights failing to be Beurling–Domar. These classes of weights are known to arise in the study of the associated Beurling algebra

$$\ell^1(S, \omega) := \left\{f : S \to \mathbb{C} : \|f\|_\omega := \sum_{s \in S} |f(s)|\omega(s) < \infty\right\}$$

with the norm $\|f\|_\omega$, which is a convolution Banach algebra [11, 14] as well as in the study of the group algebras $L^1(G, \omega)$ on a locally compact group G [10, 21]. Notice that recently the semigroup algebras $\ell^1(S)$ as well as the Beurling algebra $L^1(G, \omega)$ have acquired considerable attention [12, 13]. The investigation of interdependence between the Banach algebra structure of $\ell^1(S, \omega)$ and the structure of (S, ω) is a fascinating aspect of harmonic analysis on groups and semigroups with weights. The ω-bounded generalized semicharacters on (S, ω), $\tilde{\omega}$-bounded generalized semicharacters on $(M_\omega(S), \tilde{\omega})$ as well as $\tilde{\omega}_q$-bounded generalized semicharacters on $(M_\omega(S)/S, \tilde{\omega}_q)$ determining the Gel'fand spaces of the respective Beurling algebras are determined. These give weighted semigroup analogues of the results on the multiplier algebra of a Banach algebra discussed in Chapter 1 of [19]. In analogy with weakly regular norms [8], a weight ω on S is weakly regular if for some $m, M > 0$, $m\omega(s) \leq \tilde{\omega}(L_s) \leq M\omega(s), s \in S$. For a weakly regular ω, the Beurling algebra $\ell^1(S, \omega)$ turns out to be a closed ideal in the convolution Banach algebra $\ell^1(M_\omega(S), \tilde{\omega})$; and the quotient Banach algebra $\ell^1(M_\omega(S), \tilde{\omega})/\ell^1(S, \omega)$ gets identified with the Beurling algebra $\ell^1\left(M_\omega(S)/S, \tilde{\omega}_q\right)$ on the weighted quotient semigroup $\left(M_\omega(S)/S, \tilde{\omega}_q\right)$. Finally, the results are applied to investigate the interrelations between Banach algebras $\ell^1(S, \omega)$ and $\ell^1(M_\omega(S), \tilde{\omega})$. It so happens that $\ell^1(S, \omega)$ is semisimple if and only if $\ell^1(M_\omega(S), \tilde{\omega})$ is semisimple. Notice that the semisimplicity of a Beurling algebra is an intricate matter even when S is a (non-abelian) group (see page 175 of [12], [5]). Each of the properties regularity, uniqueness of uniform norm, $*$-regularity, uniqueness of C^*-norm (the last two in the involutive case) of $\ell^1(M_\omega(S), \tilde{\omega})$ implies the corresponding property for $\ell^1(S, \omega)$. Finally we discuss several examples exhibiting the general results.

In analogy with algebras, multipliers on S constitute a process of unitization of S. Thus there is also a unitization involving adjoining identity to S defined as $S_e = S \cup \{e\}$ with $e \notin S$, which is a unital semigroup with operation $se = es = s,\ s \in S$; $ee = e$. We show that $M(S)$ and S_e are different unitizations.

2. The weighted semigroup $(M_\omega(S), \tilde{\omega})$

The following collects elementary properties of $M(S)$ and $M_\omega(S)$.

PROPOSITION 2.1

(i) *Let $T : S \to S$ be a map such that $sTt = (Ts)t$ $(s, t \in S)$. Then T is a multiplier.*
(ii) *The set $M(S)$ is a unital abelian semigroup with composition; and S is embedded in $M(S)$ via $s \mapsto L_s$ as an ideal of $M(S)$.*
(iii) *For any weight ω on S, $M_\omega(S)$ is a subsemigroup of $M(S)$ and S is an ideal in $M_\omega(S)$.*
(iv) *If S is involutive and ω is symmetric, then each of $M(S)$ and $M_\omega(S)$ are involutive and S is a $*$-ideal.*

Proof.

(i) Let $u, s, t \in S$. Then

$$u(sTt) = u((Ts)t) = (uTs)t = (Tu)st = uT(st).$$

Since S is faithful, $T(st) = sTt$ $(s, t \in S)$.
(ii) If $T \in M(S)$ and $L_s \in S$. Then $TL_s = L_{Ts} \in S$.
(iii) Let $T, U \in M_\omega(S)$. Then there exists $K, M > 0$ such that $\omega(Ts) \leq K\omega(s),\ s \in S$ and $\omega(Us) \leq M\omega(s),\ s \in S$. Let $s \in S$. Then $\omega(TUs) \leq K\omega(Us) \leq KM\omega(s)$, i.e., $TU \in M_\omega(S)$. For $s, t \in S$, $\omega(L_t(s)) = \omega(ts) \leq \omega(t)\omega(s)$. Therefore $S \subset M_\omega(S)$. Since $TL_s = L_{Ts}$, it follows that S is an ideal in $M_\omega(S)$.
(iv) Let $T \in M_\omega(S)$. Then there exists $M > 0$ such that $\omega(Ts) \leq M\omega(s),\ s \in S$. Now

$$\omega(T^*s) = \omega((Ts^*)^*) = \omega(Ts^*) \leq M\omega(s^*) = M\omega(s),\ s \in S.$$

Therefore $M_\omega(S)$ is involutive. It is easy to see that S is a $*$-ideal of $M_\omega(S)$. □

The following shows that $M_\omega(S) \neq M(S)$ is essentially a non-unital phenomenon; and that $M(S)$ and S_e are different unitizations. A semigroup S has a *finite set of relative units* [16] if there exists a finite subset F of S such that for every $s \in S$ there exists $f \in F$ such that $sf = s$.

Theorem 2.2.

(i) *$S = M(S)$ iff S is unital.*
(ii) *If S has a finite set of relative units, then $M(S) = M_\omega(S)$ for all weight ω on S.*
(iii) *There exists a weighted semigroup (S, ω) such that $M_\omega(S) \neq M(S)$.*
(iv) *There exists a semigroup S such that $S_e \neq M(S)$.*

Proof.

Theorem 2.2(i).

(i) *is straight forward.*
(ii) *Let $F = \{f_1, f_2, \ldots, f_n\}$ be a set of relative units and let $T \in M(S)$. Let $K = \max\{\omega(Tf_1), \omega(Tf_2), \ldots, \omega(Tf_n)\}$ and let $s \in S$. Then there exists $f_i \in F$ such that $s = sf_i$. Now*

$$\omega(Ts) = \omega(T(sf_i)) = \omega(sTf_i) \leq \omega(Tf_i)\omega(s) \leq K\omega(s).$$

Therefore $M_\omega(S) = M(S)$.
(iii) *Let $S = ((1, \infty), +)$, and let $\omega(s) = \mathrm{e}^{(s-2)^{-2}}$ $(1 < s < 2)$ and $\omega(s) = 1$ $(s \geq 2)$. Then $M(S) = \{L_s : s \in [0, \infty)\}$, where $L_s(t) = s + t,\ t \in S$.*
Let $0 < s_0 < 1$. If $1 < t < 2 - s_0$, then

$$\frac{\omega(L_{s_0}(t))}{\omega(t)} = \frac{\omega(s_0 + t)}{\omega(t)} = \mathrm{e}^{(s_0+t-2)^{-2}-(t-2)^{-2}} \to \infty \text{ as } t \to 2 - s_0.$$

Hence $\sup\{\frac{\omega(L_{s_0}(t))}{\omega(t)} : t \in S\} = \infty$, i.e., $L_{s_0} \notin M_\omega(S)$. Therefore $M_\omega(S) = \{L_s : s \in \{0\} \cup [1, \infty)\} \subsetneq M(S)$.
(iv) *Let $S = \{(m, n) : m, n \in \mathbb{N}\}$ with the usual addition. Let $T \in M(S)$, and let $(m_1, n_1), (m_2, n_2) \in S$. Then $T(m_1, n_1) - (m_1, n_1) = T(m_2, n_2) - (m_2, n_2)$. Therefore $T(m, n) = (m + k, n + l)$ $(m, n) \in S$ for some $k, l \in \mathbb{N} \cup \{0\}$. Denoting this T by $T_{(k,l)}$ we have $M(S) = \{(k, l) : k, l \in \mathbb{N} \cup \{0\}\}$. The adjoining identity to S gives the unital semigroup $S_e = S \cup \{(0, 0)\}$ with natural operations.* □

The (iii) above suggests to investigate appropriate conditions on a given S for which there exists a weight ω such that $M_\omega(S) \neq M(S)$. A semigroup S is *separating* [16] if $s = t$ whenever $s^2 = t^2 = st$ and $s, t \in S$; and S is an *inverse semigroup* [17] if for every $s \in S$, there exists unique $t \in S$ such that $sts = s$ and $tst = t$; we denote this unique element by s^*. An inverse semigroup is an involutive semigroup with the involution $s^* = t$. Notice that if S is separating (in particular, inverse semigroup), then S is faithful. An involutive semigroup S is $*$-*separating* if $s = t$ whenever $s^*s = t^*t = s^*t$ and $s, t \in S$.

PROPOSITION 2.3

(i) *$\tilde{\omega}$ is a weight $M_\omega(S)$.*
(ii) *S is an inverse semigroup iff $M_\omega(S)$ is an inverse semigroup.*

(iii) *Let S be involutive and ω be symmetric. Then $\tilde{\omega}$ is symmetric; and S is $*$-separating iff $M_\omega(S)$ is $*$-separating.*

(iv) *S is separating iff $M_\omega(S)$ is separating.*

(v) *There exists a semigroup S such that both S and $M(S)$ are separating; but the quotient $M(S)/S$ fails to be separating.*

Proof.

(i) By the definition of $\tilde{\omega}$, $\tilde{\omega}(T) > 0 (T \in M_\omega(S))$. Let $T, U \in M_\omega(S)$, and let $s \in S$. Then

$$\omega(TUs) \leq \tilde{\omega}(T)\omega(Us) \leq \tilde{\omega}(T)\tilde{\omega}(U)\omega(s).$$

Therefore $\tilde{\omega}(TU) \leq \tilde{\omega}(T)\tilde{\omega}(U)$.

(ii) Let S be an inverse semigroup. Let $T, U \in M_\omega(S)$, and let $s \in S$. Then

$$TT^*T(s) = TT^*T(ss^*s) = T(s)T(s)^*T(s) = T(s).$$

Hence $TT^*T = T$. Similarly $T^*TT^* = T^*$. Suppose that $U \in M_\omega(S)$ such that $TUT = T$ and $UTU = U$. Then

$$T(s) = TUT(s) = TUT(ss^*s) = T(s)U(s^*)T(s).$$

Similarly

$$U(s^*) = U(s^*)T(s)U(s^*).$$

So $T(s)^* = U(s^*)$ and hence $T(s^*)^* = U(s)$, i.e. $T^*(s) = U(s)$. Therefore $U = T^*$. So T^* is the unique element of $M_\omega(S)$ such that $TT^*T = T$ and $T^*TT^* = T^*$. Hence $M_\omega(S)$ is an inverse semigroup.

Since $L_sL_s^*L_s = L_s$, $L_s^*L_sL_s^* = L_s^*$, $L_s^* = L_{s^*}$ and S is faithful. We have $ss^*s = s$ and $s^*ss^* = s^*$. Suppose there is some $t \in S$ such that $sts = s$ and $tst = t$. Then $L_sL_tL_s = L_s$ and $L_tL_sL_t = L_t$. Since $M_\omega(S)$ is an inverse semigroup, $L_t = L_s^* = L_{s^*}$. Again since S is faithful, $t = s^*$.

(iii) Let $T \in M_\omega(S)$. Then

$$\begin{aligned}\tilde{\omega}(T^*) &= \sup\left\{\frac{\omega(T^*(s))}{\omega(s)} : s \in S\right\} = \sup\left\{\frac{\omega(T(s^*)^*)}{\omega(s)} : s \in S\right\} \\ &= \sup\left\{\frac{\omega(T(s^*))}{\omega(s)} : s \in S\right\} = \sup\left\{\frac{\omega(T(s^*))}{\omega(s^*)} : s \in S\right\} \\ &= \sup\left\{\frac{\omega(T(s))}{\omega(s)} : s \in S\right\} = \tilde{\omega}(T).\end{aligned}$$

Hence $\tilde{\omega}$ is a symmetric weight on $M_\omega(S)$

Now assume that S is $*$-separating. Let $T, U \in M_\omega(S)$. Suppose that $T^*T = U^*U = T^*U$. Then $T^*T(s^*s) = U^*U(s^*s) = T^*U(s^*s)$ $(s \in S)$. Therefore $T(s)^*T(s) = U(s)^*U(s) = T(s)^*U(s)$. Since S is $*$-separating, $T(s) = U(s)$. Therefore $T = U$. Hence $M_\omega(S)$ is $*$-separating.

Let $M_\omega(S)$ be $*$-separating. Let $s, t \in S$. Suppose that $s^*s = t^*t = s^*t$. Then $L_s^*L_s = L_t^*L_t = L_s^*L_t$. Since $M_\omega(S)$ is $*$-separating, $L_s = L_t$. Since S is faithful, $s = t$. Thus S is $*$-separating.

(iv) This is a special case of (ii) with the trivial involution on S.

(v) Let $S = ((1, \infty), +)$. Let $T \in M(S)$, and let $s, t \in S$. Then $Ts - s = Tt - t$. Therefore there exists $k \in [0, \infty)$ such that $T(s) = s + k$ $(s \in S)$. Denoting this T by T_k we have $M(S) = ([0, \infty), +)$ and $M(S)/S = \{[t], S : t \in [0, 1]\}$. Both S and $M(S)$ are separating. Now $[1] + S = [1] + [1] = S + S$ but $[1] \neq S$. □

The weight $\tilde{\omega}$ on $M_\omega(S)$ satisfies $\omega(Ts) \leq \tilde{\omega}(T)\omega(s)$ for all $s \in S$, $T \in M_\omega(S)$. Motivated by the uniform norms and the C^*-norms in Banach algebras, we call a weight ω a *uniform weight* (respectively a C^**-weight* for an involutive S) if $\omega(s^2) = \omega(s)^2,\ s \in S$ (respectively $\omega(s^*s) = \omega(s)^2,\ s \in S$). For example, $\omega(n) = \mathrm{e}^n$ $(n \in \mathbb{N})$ is a uniform weight on $\mathbb{N}$; and $\omega(m + \lambda n) = \mathrm{e}^{-m-n}$ is a C^*-weight on $S := \{m + \lambda n : m, n \in \mathbb{N}\}$, $\lambda \in \mathbb{R}\backslash\mathbb{Q}$ fixed, having involution $(m + \lambda n)^* = n + \lambda m$. In the present case, a uniform weight is a C^*-weight for the trivial involution $s^* = s$ on S. If S is involutive and ω is a C^*-weight on S, then ω is symmetric and has uniform weight. Indeed, let $s \in S$. Then $\omega(s)^2 = \omega(s^*s) \leq \omega(s^*)\omega(s)$, i.e, $\omega(s) \leq \omega(s^*)$, and so $\omega(s^*) = \omega(s)$. Also $\omega(s)^4 = \omega(s^*s)^2 = \omega((s^*s)^2) \leq \omega(s^2)\omega((s^*)^2) = \omega(s^2)^2$, i.e., $\omega(s^2) = \omega(s)^2$. A natural setting to investigate C^*-weights would be non-abelian semigroups, but that would be a different matter.

PROPOSITION 2.4

(i) *For all* $s \in S$, $\tilde{\omega}(L_s) \leq \omega(s)$; *and the equality does not hold.*

(ii) *If* ω *is a* C^**-weight or a uniform weight on* S, *then* $\tilde{\omega}(L_s) = \omega(s)$ *for all* $s \in S$; *and* $\tilde{\omega}$ *is respectively a* C^**-weight or a uniform weight on* $M_\omega(S)$.

(iii) $\tilde{\omega}(T) = \inf\{K > 0 : \omega(Ts) \leq K\omega(s), s \in S\}, T \in M_\omega(S)$.

Proof.

(i) Let $s \in S$. Then

$$\begin{aligned}\tilde{\omega}(L_s) &= \sup\left\{\frac{\omega(L_s t)}{\omega(t)} : t \in S\right\} = \sup\left\{\frac{\omega(st)}{\omega(t)} : t \in S\right\} \\ &\leq \sup\left\{\frac{\omega(s)\omega(t)}{\omega(t)} : t \in S\right\} = \omega(s).\end{aligned}$$

Let $S = (\mathbb{N}, +)$. Consider a weight $\omega(n) = 1 + n$ on $\mathbb{N}$. Let $n \in \mathbb{N}$. Then

$$\begin{aligned}\tilde{\omega}(L_n) &= \sup\left\{\frac{\omega(L_n(m))}{\omega(m)} : m \in \mathbb{N}\right\} = \sup\left\{\frac{\omega(m+n)}{\omega(m)} : m \in \mathbb{N}\right\} \\ &= \sup\left\{\frac{1+m+n}{1+m} : m \in \mathbb{N}\right\} = 1 + \frac{n}{2}.\end{aligned}$$

This shows that $\tilde{\omega}(L_n) < \omega(n),\ n \in \mathbb{N}$.

(ii) Let ω be a C^*-weight, and let $s \in S$. Then

$$\omega(s)^2 = \omega(s^*s) \leq \tilde{\omega}(L_s)\omega(s^*) = \tilde{\omega}(L_s)\omega(s).$$

Hence $\tilde{\omega}(L_s) = \omega(s), s \in S$. Since uniform weight is a C^*-weight with trivial involution, the conclusion follows.

Suppose that ω is a uniform weight. Let $T \in M_\omega(S)$. Then

$$\frac{\omega(Ts)^2}{\omega(s)^2} = \frac{\omega((Ts)^2)}{\omega(s)^2} = \frac{\omega(sT^2s)}{\omega(s)^2} \leq \frac{\omega(T^2s)}{\omega(s)}.$$

Hence $\tilde{\omega}(T)^2 \leq \tilde{\omega}(T^2) \leq \tilde{\omega}(T)^2$, i.e., $\tilde{\omega}(T^2) = \tilde{\omega}(T)^2$.

Suppose that ω is a C^*-weight. Let $T \in M_\omega(S)$. Then

$$\frac{\omega(Ts)^2}{\omega(s)^2} = \frac{\omega((Ts)(Ts)^*)}{\omega(s)^2} = \frac{\omega((Ts)(T^*s^*))}{\omega(s)^2} = \frac{\omega(s^*(T^*Ts)}{\omega(s)^2} \leq \frac{\omega(T^2s)}{\omega(s)}.$$

Hence $\tilde{\omega}(T)^2 \leq \tilde{\omega}(T^*T) \leq \tilde{\omega}(T^*)\tilde{\omega}(T) = \tilde{\omega}(T)^2$, i.e., $\tilde{\omega}(T^*T) = \tilde{\omega}(T)^2$.

(iii) Let $T \in M_\omega(S)$, and let

$$\underline{\omega}(T) = \inf\{K > 0 : \omega(Ts) \leq K\omega(s),\ s \in S\}.$$

Since $\omega(Ts) \leq \tilde{\omega}(T)\omega(s),\ s \in S$, it follows that $\underline{\omega}(T) \leq \tilde{\omega}(T)$.

Take any $\epsilon > 0$. Then there exists $K > 0$, $\underline{\omega}(T) \leq K < \underline{\omega}(T) + \epsilon$ such that $\omega(Ts) \leq K\omega(s),\ s \in S$. Hence $\frac{\omega(Ts)}{\omega(s)} \leq K,\ s \in S$. This implies that $\tilde{\omega}(T) \leq K$. Therefore $\tilde{\omega}(T) \leq \underline{\omega}(T) + \epsilon$. Since $\epsilon > 0$ is arbitrary, $\tilde{\omega}(T) \leq \underline{\omega}(T)$. □

The following classes of weights arise in the study of associated Beurling algebras (see §3).

DEFINITION 2.5

Let ω be a weight on S. Then ω is

(i) semisimple [11] if $\lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}} > 0,\ s \in S$.
(ii) radical [11] if $\lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}} = 0,\ s \in S$.
(iii) Beurling–Domar [14] if $\omega \geq 1$ and $\sum_{n\in\mathbb{N}} \frac{\log \omega(s^n)}{1+n^2} < \infty,\ s \in S$.
(iv) GRS [15] if $\lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}} = 1,\ s \in S$.

Thus a Beurling–Domar weight is a GRS-weight, and there exists a GRS-weight which is not a Beurling–Domar weight. Indeed, let $S = ([2, \infty), +)$, and let $\omega(n) = \mathrm{e}^{\frac{n}{\log n}}$, $n \in S$. Then ω is a GRS-weight but it is not a Beurling–Domar weight.

Theorem 2.6.

(i) *ω is semisimple on S iff $\tilde{\omega}$ is semisimple on $M_\omega(S)$. If ω is a uniform weight or a C^*-weight on S, then ω is a semisimple weight on S.*
(ii) *If $\tilde{\omega}$ is a Beurling–Domar weight on $M_\omega(S)$, then ω is a Beurling–Domar weight on S.*
(iii) *Let (S, ω) satisfy any of the following conditions*:

  (a) *For each $T \in M_\omega(S)$, there exists $m \in \mathbb{N}$ such that $T^m \in S$.*
  (b) *Every element of S is idempotent.*

  If ω is a Beurling–Domar weight, then $\tilde{\omega}$ is a Beurling–Domar weight.

(iv) *Let ω be semisimple. Then* $\nu_\omega(s) := \lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}}, s \in S$ *is a uniform weight, and it is the largest uniform weight dominated by* ω.

(v) *Let ω be semisimple. Then* $\mu_\omega(s) = \nu_\omega(s^*s)^{\frac{1}{2}},\ s \in S$ *is a* C^**-weight, and it is the largest* C^**-weight dominated by* ω.

Proof.

(i) Let $T \in M_\omega(S)$, and let $s \in S$. Then $\omega(T^n(s^n)) \leq \tilde{\omega}(T^n)\omega(s^n)$. Therefore $\omega((Ts)^n) \leq \tilde{\omega}(T^n)\omega(s^n)$. This gives $\omega((Ts)^n)^{\frac{1}{n}} \leq \tilde{\omega}(T^n)^{\frac{1}{n}}\omega(s^n)^{\frac{1}{n}}$. Since ω is a semisimple weight on S, it follows that $\lim_{n\to\infty} \tilde{\omega}(T^n)^{\frac{1}{n}} > 0$.

Let $s \in S$. Since $\tilde{\omega}(L_s) \leq \omega(s)$, $\tilde{\omega}(L_s^n) = \tilde{\omega}(L_{s^n}) \leq \omega(s^n), n \in \mathbb{N}$. Since $\tilde{\omega}$ is a semisimple weight on $M_\omega(S)$, it follows that ω is a semisimple weight on S.

If ω is a uniform weight on S, then $\omega(s) = \lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}} > 0, s \in S$. Since every C^*-weight is a uniform weight, the conclusion follows.

(ii) Let $s \in S$. Then $\omega(s^{n+1}) \leq \tilde{\omega}(L_s^n)\omega(s)$. So, $\log\omega(s^{n+1}) \leq \log\tilde{\omega}(L_s^n) + \log\omega(s)$. Since $\sum_{n\in\mathbb{N}} \frac{\log\tilde{\omega}(L_s^n)}{1+n^2}$ and $\sum_{n\in\mathbb{N}} \frac{\log\omega(s)}{1+n^2}$ are finite and $\omega \geq 1$, it follows that $\sum_{n\in\mathbb{N}} \frac{\log\omega(s^{n+1})}{1+n^2} < \infty$. Since $\frac{1}{1+n^2} \geq \frac{1}{1+(n+1)^2}$, we have $\sum_{n\geq 2} \frac{\log\omega(s^n)}{1+n^2} < \infty$. Therefore ω is a Beurling–Domar weight on S.

(iii)

(a) Let $T \in M_\omega(S)$. Then there exists $m \in \mathbb{N}$ such that $T^m \in S$. Therefore $T^m = L_t$ for some $t \in S$. Then

$$\begin{aligned}
\sum_n \frac{\log\tilde{\omega}(T^n)}{1+n^2} &= \sum_{j=0}^{m-1}\sum_k \frac{\log\tilde{\omega}(T^{km+j})}{1+(km+j)^2} \\
&\leq \sum_{j=0}^{m-1}\sum_k \frac{\log\tilde{\omega}(T^{km})\tilde{\omega}(T^j)}{1+(km+j)^2} \\
&= \sum_{j=0}^{m-1}\sum_k \frac{\log\tilde{\omega}(L_t^k)\tilde{\omega}(T^j)}{1+(km+j)^2} \\
&= \sum_{j=0}^{m-1}\sum_k \frac{\log\tilde{\omega}(L_{t^k})\tilde{\omega}(T^j)}{1+(km+j)^2} \\
&\leq \sum_{j=0}^{m-1}\sum_k \frac{\log\omega(t^k)\tilde{\omega}(T^j)}{1+(km+j)^2} \\
&\leq \sum_{j=0}^{m-1}\sum_k \frac{\log\omega(t^k)\tilde{\omega}(T^j)}{1+k^2} < \infty.
\end{aligned}$$

(b) Let $T \in M_\omega(S)$, and let $s \in S$. Then $T^2s = T^2(s^2) = (Ts)^2 = Ts$. Therefore every element of $M_\omega(S)$ is idempotent. Hence for any $T \in M_\omega(S)$,

$$\sum_{n\in\mathbb{N}} \frac{\log\tilde{\omega}(T^n)}{1+n^2} = \sum_{n\in\mathbb{N}} \frac{\log\tilde{\omega}(T)}{1+n^2} < \infty.$$

(iv) Since ω is semisimple, $\nu_\omega > 0$. Let $s, t \in S$. Then

$$\begin{aligned}\nu_\omega(st) &= \lim_{n\to\infty} \omega((st)^n)^{\frac{1}{n}} \leq \lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}} \omega(t^n)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}} \lim_{n\to\infty} \omega(t^n)^{\frac{1}{n}} = \nu_\omega(s)\nu_\omega(t).\end{aligned}$$

Now

$$\nu_\omega(s^2) = \lim_{n\to\infty} \omega(s^{2n})^{\frac{1}{n}} = \lim_{n\to\infty} \omega(s^{2n})^{\frac{2}{2n}} = \nu_\omega(s)^2.$$

Hence r_ω is a uniform weight.

Let η be a uniform weight on S dominated by ω. Let $s \in S$ and $n \in \mathbb{N}$. Then

$$\eta(s) = \eta(s^{2^n})^{\frac{1}{2^n}} \leq \omega(s^{2^n})^{\frac{1}{2^n}}.$$

Hence $\eta(s) \leq \nu_\omega(s), s \in S$.

(v) Let $s, t \in S$. Then

$$\mu_\omega(st) = \nu_\omega(s^*st^*t)^{\frac{1}{2}} \leq \nu_\omega(s^*s)^{\frac{1}{2}} \nu_\omega(t^*t)^{\frac{1}{2}} = \mu_\omega(s)\mu_\omega(t).$$

Also,

$$\mu_\omega(s^*s) = \nu_\omega((s^*s)^2)^{\frac{1}{2}} = \nu_\omega(s^*) = \mu_\omega(s)^2.$$

Let ζ be a C^*-weight dominated by ω, and let $s \in S$. Then

$$\zeta(s)^2 = \zeta(s^*s) = \zeta((s^*s)^{2^n})^{\frac{1}{2^n}} \leq \omega((s^*s)^{2^n})^{\frac{1}{2^n}}.$$

Therefore $\zeta(s)^2 \leq \nu_\omega(s^*s) = \mu_\omega(s)^2$, i.e., $\zeta \leq \mu_\omega$. □

3. The Beurling algebra $\ell^1(M_\omega(S), \tilde{\omega})$

Motivated by regular and weakly regular norms on normed algebra [8], we call a weight ω on S *regular* if $\tilde{\omega}$ restricted to S is ω. More generally, ω is *weakly regular* if for some $m > 0$, $M > 0$, $m\omega(s) \leq \tilde{\omega}(L_s) \leq M\omega(s)$, $s \in S$. The weight $\omega_1(s) = e^s$ on $\mathbb{N}$ is regular; whereas $\omega_2(s) = 1 + s$ on $\mathbb{N}$ is weakly regular and non-regular.

Let ω be a weight on S such that $\omega_0 := \inf\{\omega(s) : s \in S\} > 0$, and let $\tilde{\omega}_q : M_\omega(S)/S \to (0, \infty)$ be defined as $\tilde{\omega}_q([T]) = 1$, $T \in S$ and $\tilde{\omega}_q([T]) = \tilde{\omega}(T)$, $T \notin S$. Then $\tilde{\omega}_q$ is a weight on $M_\omega(S)/S$. Indeed, let $[U], [T] \in M_\omega(S)/S$. Then $\omega_0 \leq \omega(Ts) \leq \tilde{\omega}(T)\omega(s)$ for all $s \in S$. Therefore $\tilde{\omega}(T) \geq 1$. If $UT \in S$, then $\tilde{\omega}_q([UT]) = 1 \leq \tilde{\omega}(U)\tilde{\omega}(T) = \tilde{\omega}_q(U)\tilde{\omega}_q(T)$. If $UT \notin S$, then $\tilde{\omega}_q(UT) = \tilde{\omega}(UT) \leq \tilde{\omega}(U)\tilde{\omega}(T) = \tilde{\omega}_q(U)\tilde{\omega}_q(T)$. The following exhibits the relationship between the Beurling algebras $\ell^1(S, \omega)$ and $\ell^1(M_\omega(S), \tilde{\omega})$.

Theorem 3.1. *Let ω be weakly regular with $\omega_0 > 0$. Then $\ell^1(S, \omega)$ is a closed ideal of $\ell^1(M_\omega(S), \tilde{\omega})$ and the quotient algebra $\ell^1(M_\omega(S), \tilde{\omega})/\ell^1(S, \omega)$ is isomorphic to the Beurling algebra of the Rees quotient semigroup $M_\omega(S)/S$ with the quotient weight $\tilde{\omega}_q$.*

Proof. Let $f = \sum_{s\in S} f(s)\delta_s \in \ell^1(S,\omega)$. Then we can write f as $f = \sum_{s\in S} f(L_s)\delta_{L_s}$. Now

$$\sum_{s\in S} |f(L_s)|\tilde{\omega}(L_s) \leq \sum_{s\in S} |f(L_s)|\omega(s) < \infty.$$

Therefore $\ell^1(S,\omega) \subset \ell^1(M_\omega(S),\tilde{\omega})$. Since ω is weakly regular, $\ell^1(S,\omega)$ is a closed subalgebra of $\ell^1(M_\omega(S),\tilde{\omega})$.

If $F = \sum_{T\in M_\omega(S)} F(T)\delta_T \in \ell^1(M_\omega(S),\tilde{\omega})$ and $f = \sum_{s\in S} f(L_s)\delta_{L_s} \in \ell^1(S,\omega)$. Then

$$\begin{aligned}
&\left\| \left(\sum_{T\in M_\omega(S)} F(T)\delta_T \right) \star \left(\sum_{s\in S} f(L_s)\delta_{L_s} \right) \right\|_\omega \\
&= \left\| \sum_{T\in M_\omega(S)} \sum_{s\in S} f(L_s)F(T)\delta_{L_{Ts}} \right\|_\omega \\
&= \sum_{T\in M_\omega(S)} \sum_{s\in S} |f(L_s)F(T)|\, \omega(Ts) \\
&\leq \sum_{T\in M_\omega(S)} \sum_{s\in S} |f(L_s)F(T)|\, \tilde{\omega}(T)\omega(s) \\
&= \left(\sum_{T\in M_\omega(S)} |F(T)|\tilde{\omega}(T) \right) \left(\sum_{s\in S} |f(L_s)|\omega(s) \right) \\
&= \|F\|_{\tilde{\omega}} \|f\|_\omega.
\end{aligned}$$

Therefore $\ell^1(S,\omega)$ is an ideal in $\ell^1(M_\omega(S),\tilde{\omega})$.

Since ω is weakly regular, any element F of $\ell^1(M_\omega(S),\tilde{\omega})$ can be written as $F = F_0 + F_1$, where $F_0 \in \ell^1(S,\omega)$ and $F_1 \in \ell^1(M_\omega(S)\setminus S,\tilde{\omega})$. Define $\varphi : \ell^1\,(M_\omega(S),\tilde{\omega}) \to \ell^1\left(M_\omega(S)/S, \tilde{\omega}_q\right)$ as

$$\varphi(F) = F_1 \quad (F \in \ell^1\,(M_\omega(S),\tilde{\omega})).$$

Here we have identified the elements of $\ell^1(M_\omega\setminus S,\tilde{\omega})$ with the elements of $\ell^1\left(M_\omega(S)/S,\tilde{\omega}_q\right)$. It is clear that $F \in \ker\varphi$ iff $F_1 = 0$. Hence $\ker\varphi = \ell^1(S,\omega)$. □

To describe the Gel'fand spaces of the Banach algebras involved, we consider generalized semicharacters. A *generalized semicharacter* on S is a non-zero map $\alpha : S \to \mathbb{C}$ satisfying $\alpha(st) = \alpha(s)\alpha(t),\ \ s,t \in S$. An *$\omega$-bounded generalized semicharacter* on (S,ω) is a generalized semicharacter on S satisfying $|\alpha(s)| \leq \omega(s), s \in S$. Let $\Phi_{\omega s}(S)$ denote the set of all ω-bounded generalized semicharacters on S with the point open topology. Let ω be a symmetric weight on a $*$-semigroup S, and let α be a generalized semicharacter on S. The *adjoint* α^* of α is a map on S defined as $\alpha^*(s) = \overline{\alpha(s^*)}, s \in S$. Then α^* is a generalized semicharacter on S. A generalized semicharacter on S is *self adjoint* if $\alpha = \alpha^*$. Let $\Psi_{\omega s}(S)$ denote the set of all self adjoint generalized semicharacters on S with the point open topology. Our next three results contain the semigroup multiplier analogues of a couple of results on multipliers on commutative Banach algebras (Theorems 1.4.1, 1.4.2, Corollary 1.4.1 of [19]).

Theorem 3.2. *If $\alpha \in \Phi_{\omega s}(S)$, then there exists an unique $\tilde{\alpha} \in \Phi_{\tilde{\omega}s}(M_\omega(S))$ such that $\tilde{\alpha}(L_s) = \alpha(s)$ for all $s \in S$. If $\beta \in \Phi_{\tilde{\omega}s}(M_\omega(S))$, then either $\beta(L_s) = 0$ for all $s \in S$ or there is $\tilde{\alpha} \in \Phi_{\tilde{\omega}s}(M_\omega(S))$ such that $\beta = \tilde{\alpha}$.*

Proof. Let $\alpha \in \Phi_{\omega s}(S)$, and let $s \in S$ be such that $\alpha(s) \neq 0$. Define

$$\tilde{\alpha}(T) = \frac{\alpha(Ts)}{\alpha(s)}, \qquad T \in M(S).$$

This definition is independent of the choice of s. Since, if $t \in S$ such that $\alpha(t) \neq 0$, then

$$\begin{aligned}\tilde{\alpha}(T)\alpha(t) &= \frac{\alpha(Ts)}{\alpha(s)}\alpha(t) = \frac{\alpha(Tst)}{\alpha(s)} = \frac{\alpha(sTt)}{\alpha(s)} \\ &= \alpha(s)\frac{\alpha(Tt)}{\alpha(s)} = \alpha(Tt).\end{aligned}$$

Also $\tilde{\alpha}(L_t) = \frac{\alpha(L_t s)}{\alpha(s)} = \frac{\alpha(st)}{\alpha(s)} = \frac{\alpha(s)\alpha(t)}{\alpha(s)} = \alpha(t)$ for all $t \in S$. Moreover, if $T, U \in M(S)$, then

$$\tilde{\alpha}(TU) = \frac{\alpha(TUs)}{\alpha(s)} = \frac{\tilde{\alpha}(T)\alpha(Us)}{\alpha(s)} = \tilde{\alpha}(U)\tilde{\alpha}(T).$$

Thus $\tilde{\alpha}$ is multiplicative on $M(S)$. Now we prove that $\tilde{\alpha} \in \Phi_{\tilde{\omega}s}(M_\omega(S))$. Let $T \in M_\omega(S)$. Then

$$|\alpha(s)||\tilde{\alpha}(T)| = |\alpha(Ts)| \leq \omega(Ts) \leq \tilde{\omega}(T)\omega(s).$$

Therefore for any $n \in \mathbb{N}$,

$$|\alpha(s)||\tilde{\alpha}(T)|^n = |\alpha(s)||\tilde{\alpha}(T^n)| \leq \tilde{\omega}(T^n)\omega(s) \leq \tilde{\omega}(T)^n\omega(s).$$

This implies that

$$|\alpha(s)|^{\frac{1}{n}}|\tilde{\alpha}(T)| \leq \tilde{\omega}(T)\omega(s)^{\frac{1}{n}}.$$

Hence $|\tilde{\alpha}(T)| \leq \tilde{\omega}(T)$. Finally, $\tilde{\alpha}$ is unique. As if $\alpha^* \in \Phi_{\tilde{\omega}s}(M_\omega(S))$ such that $\alpha^*(L_s) = \alpha(s)$ $(s \in S)$, then choosing $s \in S$ for which $\alpha(s) \neq 0$, we see that for each $T \in M_\omega(S)$,

$$\begin{aligned}\alpha^*(T)\alpha(s) &= \alpha^*(T)\alpha^*(L_s) = \alpha^*(TL_s) = \alpha^*(L_{Ts}) \\ &= \alpha(Ts) = \tilde{\alpha}(T)\alpha(s).\end{aligned}$$

Hence $\alpha^* = \tilde{\alpha}$.

On the other hand, if $\beta \in \Phi_{\tilde{\omega}s}(M_\omega(S))$ and $\beta(L_s) \neq 0$ for some $s \in S$, then clearly the equation $\alpha(s) = \beta(L_s)$, $s \in S$ defines an element of $\Phi_{\omega s}(S)$. From the first portion of the proof it then follows that $\beta = \tilde{\alpha}$. □

Let $\tilde{\Phi}_{\tilde{\omega}s}(S) = \{\tilde{\alpha} \in \Phi_{\tilde{\omega}s}(M_\omega(S)) : \tilde{\alpha}(L_s) \neq 0 \text{ for some } s \in S\}$, and let $h_{\omega s}(S) = \{\tilde{\alpha} \in \Phi_{\tilde{\omega}s}(M_\omega(S)) : \tilde{\alpha}(L_s) = 0, s \in S\}$. The previous result asserts that the correspondence $\alpha \mapsto \tilde{\alpha}$ defines a bijective mapping of $\Phi_{\omega s}(S)$ onto those points of $\Phi_{\tilde{\omega}s}(M_\omega(S))$ which do not contain the ideal $\{L_s : s \in S\}$, that is, those ω-bounded generalized semicharacters on $M_\omega(S)$ which do not vanish identically on $\{L_s : s \in S\}$. We shall denote this subset of $\Phi_{\tilde{\omega}s}(M_\omega(S))$ by $\tilde{\Phi}_{\omega s}(S)$.

COROLLARY 3.3

Let ω be a weight on a semigroup S.

(i) *Then $\Phi_{\tilde{\omega}s}(M_\omega(S)) = \tilde{\Phi}_{\omega s}(S) \cup h_{\omega s}(S)$.*
(ii) *Let S be involutive, and let ω be symmetric. Let $h^*_{\omega s}(S) = \{\alpha \in \Psi_{\omega s}(M_\omega(S)) : \alpha(S) = \{0\}\}$. Then $\Psi_{\tilde{\omega}s}(M_\omega(S)) = \tilde{\Psi}_{\omega s}(S) \cup h^*_{\omega s}(S)$.*

Proof.

(i) From Theorem 3.2 and the preceding remarks, it is apparent that $\Phi_{\tilde{\omega}s}(M_\omega(S)) = \tilde{\Phi}_{\omega s}(S) \cup h_{\omega s}(S)$.
(ii) Let $\psi \in \Psi_{\omega s}(S)$ and $T \in M_\omega(S)$. Let $s \in S$ be such that $\psi(s) \neq 0$. Then

$$\begin{aligned} \psi'^*(T) &= \overline{\psi'(T^*)} = \overline{\left(\frac{\psi(T^*s)}{\psi(s)}\right)} = \overline{\left(\frac{\psi((Ts^*)^*)}{\psi(s)}\right)} \\ &= \frac{\psi^*(Ts^*)}{\psi^*(s^*)} = \frac{\psi(Ts^*)}{\psi(s^*)} = \psi'(T). \end{aligned}$$

Conversely, let $\psi' \in \Psi_{\omega s}(M_\omega(S))$ and $\psi'(L_s) \neq 0$ for some $s \in S$. Then

$$\psi^*(s) = \overline{\psi(s^*)} = \overline{\psi'(L_s^*)} = \psi'^*(L_s) = \psi(s).$$

The rest is similar to (i). □

The following corresponds to the result that for a weighted locally compact abelian group (G, ω), the Gel'fand space $\Delta(L^1(G, \omega))$ is identified with the space of ω-bounded generalized characters on G. We omit the straightforward proof.

COROLLARY 3.4

(i) $\Delta(\ell^1(S, \omega)) \cong \Phi_{\omega s}(S)$, *topologically as well.*
(ii) $\Delta(\ell^1(M_\omega(S), \tilde{\omega})) \cong \tilde{\Phi}_{\omega s}(S) \cup h_{\omega s}(S)$.
(iii) *Let ω be weakly regular. Then* $\Delta\left(\ell^1\left(M_\omega(S)/S, \tilde{\omega}_q\right)\right) \cong h_{\omega s}(S)$.

Semisimplicity of a Beurling algebra is an important problem. For a locally compact group G, $L^1(G, \omega)$ is semisimple if G is abelian [5]; for non-abelian G, it is not known whether $L^1(G, \omega)$ is semisimple or not (page 175 of [12]). For an abelian semigroup S, $\ell^1(S, \omega)$ is semisimple iff S is separating and ω is semisimple (Proposition 4.8 of [11]). This quickly gives the following.

Theorem 3.5. *The Banach algebra $\ell^1(S, \omega)$ is semisimple iff $\ell^1(M_\omega(S), \tilde{\omega})$ is semisimple. The quotient $\ell^1(M_\omega(S), \tilde{\omega})/\ell^1(S, \omega)$ may fail to be semisimple.*

Proof. Let $S = (\mathbb{N} \setminus \{1\}, +)$, and let $\omega(n) = 1 + n^2$ $(n \in S)$. Then $M_\omega(S) = \{T_n : n \in \mathbb{N} \cup \{0\}\}$, where $T_n(m) = n + m$ $(m \in S)$, and $\tilde{\omega}(T_n) = \frac{1+(n+2)^2}{5}$ $(n \in \mathbb{N} \cup \{0\})$. Here both

$\ell^1(S,\omega)$ and $\ell^1(M_\omega(S),\tilde{\omega})$ are semisimple. The quotient $M_\omega(S)/S = \{\bar{0},\bar{1},S\}$, which is not separating. So $\ell^1(M_\omega(S)/S,\tilde{\omega}_q)$ is not semisimple. □

In the context of the not necessarily commutative Banach $*$-algebras, the properties uniqueness of C^*-norm (UC*NP) and $*$-regularity (Section 10.5 of [20]) have been introduced and investigated by Barnes [2]; and have found applications in non-commutative harmonic analysis (see references in [20]). Analogously, in the context of commutative, not-necessarily involutive Banach algebras, uniqueness of uniform norm property (UUNP) is introduced in [7]. A Banach algebra $(\mathcal{A},\|\cdot\|)$ has UUNP if it admits exactly one uniform norm, not necessarily complete. A *uniform norm* on a Banach algebra $(\mathcal{A},\|\cdot\|)$ is a norm $|\cdot|$ satisfying $|x^2| = |x|^2$, $x \in \mathcal{A}$. The UUNP turns out to be closely related with regularity [6, 18] and have applications to abelian harmonic analysis [3, 4, 9]. Recall that $\mathcal{A}$ is *regular* if in the Gel'fand space $\Delta(\mathcal{A})$, a point and a closed set can be separated by a Gel'fand transform [18]. For an abelian G, the algebra $\ell^1(G)$ is regular; and for a weighted group G, $\ell^1(G,\omega)$ is regular iff $\ell^1(G,\omega)$ has UUNP iff ω is a Beurling–Domar weight [6]. It would be interesting to search for a weighted semigroup (S,ω) such that $\ell^1(S,\omega)$ has UUNP but is not regular.

Theorem 3.6.

(i) *If $\ell^1(M_\omega(S),\tilde{\omega})$ has UUNP, then $\ell^1(S,\omega)$ has UUNP.*
(ii) *If $\ell^1(M_\omega(S),\tilde{\omega})$ is regular, then $\ell^1(S,\omega)$ is regular.*
(iii) *Let S be an inverse semigroup. Let ω be a Beurling–Domar weight on S. Then $\ell^1(S,\omega)$ is regular.*

Proof.

(i) In view of the fact that a Banach algebra has either exactly one uniform norm or infinitely many uniform norms (Theorem 2.2(i) of [9]), it is enough to show that if $\ell^1(S,\omega)$ has infinitely many uniform norms, then so does $\ell^1(M_\omega(S),\tilde{\omega})$. Let $|\cdot|$ be a uniform norm on $\ell^1(S,\omega)$. Then there exists a closed subset F of $\Delta(\ell^1(S,\omega))$ such that $|\cdot| = |\cdot|_F$ on $\ell^1(S,\omega)$, where $|x|_F := \sup\{|\varphi(x)| : \varphi \in F\}$. Let $\tilde{F} = F \cup h_{\omega s}(S)$. Define

$$|T|_{\tilde{F}} = \sup\{|\varphi(T)| : \varphi \in \tilde{F}\}, \qquad T \in \ell^1(M_\omega(S),\tilde{\omega}).$$

It is clear that $|\cdot|_{\tilde{F}}$ is a uniform seminorm on $\ell^1(M_\omega(S),\tilde{\omega})$ and $|\cdot|_{\tilde{F}} = |\cdot|_F$ on $\ell^1(S,\omega)$. Suppose that $|T|_{\tilde{F}} = 0$. Then $\hat{T}|_{h_{\omega s}(S)} = 0$ and $\hat{T}|_F = 0$. Let $\psi \in \Delta(\ell^1(S,\omega)) \setminus F$. Then there exists $f \in \ell^1(S,\omega)$ such that $\psi(f) = 1$. Since $\ell^1(S,\omega)$ is an ideal in $\ell^1(M_\omega(S),\tilde{\omega})$, $fT \in \ell^1(S,\omega)$. Now

$$\widehat{(fT)}(\varphi) = \hat{f}(\varphi)\hat{T}(\varphi) = 0 \ (\varphi \in F).$$

Therefore $|fT| = 0$. Since $|\cdot|$ is a norm on $\ell^1(S,\omega)$, $fT = 0$. So

$$0 = \widehat{(fT)}(\psi) = \hat{f}(\psi)\hat{T}(\psi) = 0.$$

Thus $\hat{T}|_{\Delta(\ell^1(S,\omega))} = 0$ and $\hat{T}|_{h_{\omega s}(S)} = 0$. So $\hat{T} = 0$. Since $\ell^1(M_\omega(S),\tilde{\omega})$ is semisimple, $T = 0$. Thus $|\cdot|_{\tilde{F}}$ is a uniform norm on $\ell^1(M_\omega(S),\tilde{\omega})$.

(ii) Let F be a closed subset of $\Delta(\ell^1(S,\omega))$ and $\varphi \in \Delta(\ell^1(S,\omega))\setminus F$. Let $\tilde{F} = F \cup h_{\omega s}(S)$. Then $\tilde{F}$ is a closed subset of $\Delta(\ell^1(M_\omega(S),\tilde{\omega}))$ and $\varphi \in \Delta(\ell^1(M_\omega(S),\tilde{\omega}))\setminus \tilde{F}$. Since $\ell^1(M_\omega(S),\tilde{\omega})$ is regular, there exists $T \in \ell^1(M_\omega(S),\tilde{\omega})$ such that $\hat{T}(\tilde{F}) = \{0\}$ and $\hat{T}(\varphi) = 1$. There exists $f \in \ell^1(S,\omega)$ such that $\hat{f}(\varphi) = 1$. Since $\ell^1(S,\omega)$ is an ideal of $\ell^1(M_\omega(S),\tilde{\omega})$, $fT \in \ell^1(S,\omega)$. Let $\psi \in F$. Then $\widehat{fT}(\psi) = \hat{f}(\psi)\hat{T}(\psi) = 0$ and $\widehat{fT}(\varphi) = \hat{f}(\varphi)\hat{T}(\varphi) \neq 0$, which shows that $\ell^1(S,\omega)$ is regular.

(iii) As S is an abelian inverse semigroup, it can be written as a disjoint union of groups, say

$$S = \bigcup_e G_e.$$

Now $\ell^1(G_e,\omega)$ is regular. By [1], $\ell^1(S,\omega)$ has the largest closed subalgebra, say $\operatorname{Reg}\ell^1(S,\omega)$, which is regular. Then $\bigcup_e \ell^1(G_e,\omega) \subseteq \operatorname{Reg}\ell^1(S,\omega)$. Therefore $\overline{\operatorname{sp}\bigcup_e \ell^1(G_e,\omega)} \subseteq \operatorname{Reg}\ell^1(S,\omega)$. Let $f \in C_c(S)$. Then $f = f_1 + f_2 + \cdots + f_n$, where $f_i \in \ell^1(G_i,\omega), 1 \le i \le n$. Therefore $f \in \operatorname{Reg}\ell^1(S,\omega)$. This implies that $C_c(S) \subseteq \operatorname{Reg}\ell^1(S,\omega)$. Thus $\operatorname{Reg}\ell^1(S,\omega) = \ell^1(S,\omega)$. Hence $\ell^1(S,\omega)$ is regular. □

A Banach $*$-algebra $(\mathcal{B}, \|\cdot\|)$ has *unique C^*-norm property* (*UC*NP*) [2] if $\mathcal{B}$ admits exactly one C^*-norm. A commutative Banach $*$-algebra $\mathcal{B}$ is *$*$-regular* [2] if given $F \subset \tilde{\Delta}(\mathcal{B})$ closed and $\varphi \notin F$, there exists $x \in \mathcal{B}$ such that $\hat{x}(\varphi) \neq 0$ and $\hat{x}(F) = \{0\}$. In fact, UC^*NP and $*$-regularity (appropriately defined) acquires much greater significance in non-commutative Banach $*$-algebras [2]. Their role in commutative Banach $*$-algebras is discussed in Section 2 of [2], [6, 9]. By [15], for a weighted compactly generated (not necessarily abelian) group (G,ω), $L^1(G,\omega)$ is symmetric iff ω is a GRS-weight. By [2], a commutative Banach $*$-algebra is regular iff it is $*$-regular and symmetric.

Theorem 3.7. *Let S be involutive, and let ω be symmetric.*

(i) *If $\ell^1(M_\omega(S),\tilde{\omega})$ has UC*NP, then $\ell^1(S,\omega)$ has UC*NP.*
(ii) *If $\ell^1(M_\omega(S),\tilde{\omega})$ is $*$-regular, then $\ell^1(S,\omega)$ is $*$-regular.*

The proof is exactly analogous to that of Theorem 3.6, in the present case, based on Theorem 2.2(ii) of [9] which states that a commutative Banach $*$-algebra admits either exactly one C^*-norm or infinitely many C^*-norms.

It is tempting to conjecture that each of the properties UUNP and regularity of $\ell^1(S,\omega)$ implies the corresponding property for $\ell^1(M_\omega(S),\tilde{\omega})$ provided it also holds for the quotient $\ell^1\left(M_\omega(S)/S, \tilde{\omega}_q\right)$. Another closely related issue is whether the multiplier algebra $M(\ell^1(S,\omega))$ is isomorphic to the Beurling algebra $\ell^1(M_\omega(S),\tilde{\omega})$ of the weighted multiplier semigroup $(M_\omega(S),\tilde{\omega})$ (even when $\omega \equiv 1$). Notice that $\ell^1(M(S)) \subset M(\ell^1(S))$.

4. Examples

(i) Let $S = (\mathbb{N}, +)$, and let $\omega(n) = 1 + n$ $(n \in S)$. Then $M_\omega(S) = \{T_n : n \in \mathbb{N} \cup 0\}$, where $T_n(m) = n + m, m \in S$ and $\tilde{\omega}(T_n) = 1 + \frac{n}{2}, n \in \mathbb{N} \cup \{0\}$. Then ω is semisimple; ω is not a uniform weight and hence not a C^*-weight; ω is weakly regular but not regular; $\ell^1(S,\omega)$ does not have UC^*NP so does not have UUNP. Hence $\ell^1(M_\omega(S),\tilde{\omega})$ does not have UC^*NP and so does not have UUNP; $\Phi_{\omega s}(S) = \{z \in \mathbb{C} : |z| \le 1, z \ne 0\}$ and $\Phi_{\tilde{\omega} s}(M_\omega(S)) = \{z \in \mathbb{C} : |z| \le 1\}$, both via the map $z \mapsto \varphi_z, \varphi_z(n) = z^n$.

(ii) Let $S = (\mathbb{N}, +)$, and let $\omega(n) = \mathrm{e}^n, n \in S$. Then $M_\omega(S) = \{T_n : n \in \mathbb{N} \cup \{0\}\}$, where $T_n(m) = n + m, m \in S$, and $\tilde{\omega}(T_n) = \mathrm{e}^n, n \in \mathbb{N} \cup \{0\}$. Then ω is a C^*-weight and hence it is uniform and semisimple weight; ω is a regular weight; both $\ell^1(S,\omega)$ and $\ell^1(M_\omega(S),\tilde{w})$ do not have UC^*NP and hence do not have UUNP; $\Phi_{\omega s}(S) = \{z \in \mathbb{C} : |z| \le e, z \ne 0\}$ and $\Phi_{\tilde{\omega} s}(M_\omega(S)) = \{z \in \mathbb{C} : |z| \le e\}$.

(iii) Let $S = (\mathbb{N}, +)$, and let $\omega(n) = \mathrm{e}^{-n^2}, n \in S$. Then $M_\omega(S) = \{T_n : n \in \mathbb{N} \cup \{0\}\}$, where $T_n(m) = n + m, m \in S$, and $\tilde{\omega}(T_n) = \mathrm{e}^{-n^2-2n}, n \in \mathbb{N} \cup \{0\}$. Then both ω and $\tilde{\omega}$ are radical weights; $\ell^1(S,\omega)$ is not semisimple and hence $\ell^1(M_\omega(S),\tilde{\omega})$ is not semisimple; in fact, $\ell^1(S,\omega)$ is a radical Banach algebra. Both $\ell^1(S,\omega)$ and $\ell^1(M_\omega(S),\tilde{w})$ do not have UC^*NP and hence do not have UUNP; $\Phi_{\omega s}(S) = \Phi_{\tilde{\omega} s}(M_\omega(S)) = \emptyset$.

(iv) Let $S = ([1,\infty), +)$, and let $\omega(s) = \exp(-s^\gamma), s \in S$, where $0 < \gamma < 1$. Then $M_\omega(S) = \{T_s : s \in [0,\infty)\}$, where $T_s(t) = s + t, t \in S$ and $\tilde{\omega}(T_s) = \exp(1 - (s + 1)^\gamma), s \in [0,\infty)$. Then ω is semisimple and so is $\tilde{\omega}$; ω is not a uniform weight and so $\tilde{\omega}$ is also not a uniform weight (and hence they are not C^*-weights); ω is weakly regular but not regular; $\Phi_{\omega s}(S) = \{\alpha_z : \operatorname{Re} z \le -1\}$ and $\Phi_{\tilde{\omega} s}(M_\omega(S)) = \{\alpha_z : \operatorname{Re} z \le -\gamma\}$, where $\alpha_z(s) = \exp(zs)$.

(v) Let X be any set, and let $e \in X$. Define the multiplication on X as follows: $st = e$ $(s \ne t)$ and $st = s$ $(s = t)$. Then X is an abelian inverse semigroup with $s^* = s, s \in X$. We denote this semigroup by X_e.

(a) If $\omega : X_e \to (0,\infty)$ is a weight on X_e, then $\omega(s) \ge 1, s \in X$. The converse holds if $\omega(s) \ge \omega(e) \ge 1,\ s \in X$.
(b) $M(X_e) \cong \mathcal{P}(X_e) \setminus \{\emptyset\}$.
(c) Every weight on X_e is symmetric and is a Beurling–Domar weight.
(d) No non-trivial weight on X_e is a uniform weight or C^*-weight.
(e) If ω is a weight on X_e, then $\tilde{\omega} \le \omega(e)$.
(f) $\Phi_{\omega s}(X_e) = \Psi_{\omega s}(X_e) = \{\theta_t : t \in X_e\}$.
(g) $\ell^1(X_e,\omega)$ is regular.

(a) Let $s \in X_e$. Then $\omega(s) = \omega(s^2) \le \omega(s)\omega(s)$. Therefore $\omega(s) \ge 1$ $(s \in X_e)$. Conversely, let $s, t \in X_e$. Then $\omega(st)$ is either $\omega(e)$ or $\omega(s)$. Since $\omega(s) \ge 1$ $(s \in X_e)$, $\omega(st) \le \omega(s)\omega(t)$.

(b) Let $T \in M(X_e)$. Then for any $s \in X_e$, $Ts = T(s^2) = sTs$. Therefore $Ts = s$ or $Ts = e$. We also note that $Te = e$. Let $F = \{s \in S : Ts = s\}$. Then $Ts = s, s \in F$ and $Ts = e, s \notin F$. Conversely, given $\emptyset \ne F \subset X_e$, define $T : X_e \to X_e$ as $Ts = s, s \in F$ and $Ts = e, s \notin F$. Then $T \in M(X_e)$.

(c) Since $s = s^*, s \in X_e$ and $\omega(s^n) = \omega(s), s \in X_e,\ n \in \mathbb{N}$, the conclusion follows.

(d) If ω is a uniform weight, then for any $s \in X_e$ $\omega(s) = \omega(s^2) = \omega(s)^2$. Hence $\omega \equiv 1$.

(e) For any $s \in X_e$, $\omega(s) = \omega(s^2) \leq \omega(s)^2$. Hence $\omega \geq 1$. Let $T \in M_\omega(X_e)$. Then there exists $\emptyset \neq F \subset X_e$ such that $Ts = s, s \in F$ and $Ts = e, s \notin F$. Now

$$\begin{aligned} \tilde{\omega}(T) &= \sup\left\{\frac{\omega(Ts)}{\omega(s)} : s \in X_e\right\} \\ &= \sup\left\{1, \frac{\omega(e)}{\omega(s)} : s \notin F\right\} \leq \omega(e). \end{aligned}$$

(f) Let $\theta \in \Phi_{\omega s}(X_e)$. Then $\theta(e) = 0$ or 1. Let $\theta(e) = 0$. Let $t \in S$ be such that $\theta(t) = 1$. Then for any $s \in X_e \setminus \{t\}$, $0 = \theta(e) = \theta(st) = \theta(s)\theta(t)$. Hence $\theta(s) = 1, s = t$ and $\theta(s) = 0, s \neq t$. We denote this element of $\Phi_{\omega s}(X_e)$ by θ_t. Now assume that $\theta(e) = 1$. Then $\theta(s) = \theta(s)\theta(e) = \theta(se) = \theta(e) = 1$. Therefore $\theta \equiv 1$. We denote this element of $\Phi_{\omega s}(X_e)$ by θ_e. Hence $\Phi_{\omega s}(X_e) = \{\theta_s : s \in X_e\}$.

(g) Since X_e is an abelian inverse semigroup and every weight on S is a Beurling–Domar weight, it follows that for any weight ω on X_e, $\ell^1(X_e, \omega)$ is regular.

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