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A remark on the Lifshitz tail for Schrödinger operator with random magnetic field

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Abstract. In this note, we consider the Lifshitz singularity for Schrödinger operator with ergodic random magnetic field. A key estimate is an energy bound for magnetic Schrödinger operators as discussed in Nakamura [8]. Here we remove a technical assumption in [8], namely, the uniform boundedness of the magnetic field.

Keywords. Random Schrödinger operators; Lifshitz tail; random magnetic fields.

1. Introduction

We consider the magnetic Schrödinger operator without scalar potential on $\mathbb{R}^d$ ($d \geq 2$):

$$H = (p - A(x))^2 \qquad \text{on } L^2(\mathbb{R}^d),$$

where $p = -i\partial_x$ is the momentum operator,

$$A(x) = (A_1(x), \ldots, A_d(x)), \qquad x \in \mathbb{R}^d$$

is a vector potential. We suppose A is a vector-valued C^1-class function and identify $A(x)$ with the 1-form $A = \sum_{j=1}^d A_j(x)\mathrm{d}x_j$. The magnetic field is the 2-form defined by

$$B(x) = \mathrm{d}A = \sum_{i<j} B_{ij}(x)\mathrm{d}x_i \wedge \mathrm{d}x_j,$$

where $B_{ij}(x) = \partial_{x_i} A_j(x) - \partial_{x_j} A_i(x)$, and we identify B with the skew-symmetric matrix valued function $(B_{ij}(x))$ on $\mathbb{R}^d$. It is well-known that for any closed 2-form B, there exists a 1-form A such that $B = \mathrm{d}A$, and the corresponding Schrödinger operator H is uniquely determined by B modulo the gauge transformation group.

We suppose that $B = B^\omega$ is a metrically transitive random closed 2-form: namely, there exist a probability space $(\Omega, \Sigma, \mathbb{P})$ and $\mathbb{R}^d$ acts on Ω as a measure preserving transformation group. We denote the action by $\{T_x | x \in \mathbb{R}^d\}$. The action is supposed to be ergodic, i.e., if $E \subset \Omega$ is invariant under $\{T_x\}$ then $\mathbb{P}(E) = 0$ or 1. $B = B^\omega$ is assumed to satisfy

$$B^\omega(x - y) = B^{(T_x\omega)}(x), \qquad x, y \in \mathbb{R}^d$$

and it is a continuous closed 2-form almost surely. We denote

$$\Lambda_L = \{x \in \mathbb{R}^d \,||x_j| \leq L/2 \text{ for } j = 1, \ldots, d\}, \qquad L > 0$$

and let H_Λ^ω be the Schrödinger operator (with the same symbol as above) on $\Lambda \subset \mathbb{R}^d$ with the Neumann boundary condition. We also denote the Lebesgue measure of Λ by $|\Lambda|$. Then the integrated density of states

$$k(E) = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \#\{\text{eigenvalue of } H_{\Lambda_L}^\omega \le E\}$$

exists for almost all $E \in \mathbb{R}$ and $\omega \in \Omega$, and it is independent of ω (almost surely). This result is well-known for random Schrödinger operator without magnetic field (see, e.g., [3, 5]), and it is proved for magnetic Schrödinger operators in Doi, Iwatsuka, Mine [4] and Nakamura [9].

Our assumption is the following:

Assumption A. (i) B^ω is a metrically transitive random closed 2-form on $\mathbb{R}^d$ in the above sense, and it is nontrivial, namely, $\mathbb{E}(|B^\omega(0)|) \ne 0$. (ii) There exists a real-valued continuous function φ on $\mathbb{R}_+$ such that $\varphi(\lambda) \to 0$ as $\lambda \to \infty$, and it satisfies the following condition: Let $\Lambda_1, \Lambda_2 \in \Sigma$ and let $f \in L^1(\Omega)$, $g \in L^\infty(\Omega)$ such that f is Σ_{Λ_1}-measurable and g is Σ_{Λ_2}-measurable, where Σ_Λ denotes the σ-algebra generated by $\{B^\omega(x) | x \in \Lambda\}$. Then

$$|\mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g)| \le \varphi(\mathrm{dist}(\Lambda_1, \Lambda_2)) \|f\|_{L^1} \|g\|_{L^\infty},$$

where $\mathbb{E}(\cdot)$ denotes the expectation with respect to $\mathbb{P}$, and dist $(\cdot, \cdot)$ is the Euclidean distance on $\mathbb{R}^d$.

The second assumption implies that, roughly speaking, $B^\omega(x)$ and $B^\omega(y)$ are almost independent if $|x - y|$ is sufficiently large.

Theorem 1. *Suppose Assumption A. Then*

$$\limsup_{E\downarrow 0} \log(-\log k(E))/\log E \le -d/2,$$

i.e., for any $\epsilon > 0$,

$$k(E) \le \exp(-E^{-d/2+\epsilon}) \quad \textit{for sufficiently small } E \ge 0.$$

This theorem is proved in Nakamura [8] with an additional assumption: $|B^\omega(x)| \le M$ uniformly for all $\omega \in \Omega$ and $x \in \mathbb{R}^d$. We discuss the modification of the proof necessary to remove this technical assumption in § 2.

Not much rigorous results has been obtained on Schrödinger operators with random magnetic fields. We only mention a paper by Ueki [10], and one by the author [7] for the two dimensional discrete case.

The key estimate in the proof is the following energy estimate. We write it down for the two dimensional case ($d = 2$) only, in order to simplify the notations. Let $H = (p - A(x))^2$ be a magnetic Schrödinger operator, and let $B(x) = \mathrm{rot}A(x)$ be the magnetic field. For $x \in \mathbb{R}^2$, we let

$$D(x) = \{y \in \mathbb{R}^2 \big| |x - y| \le r\}, \qquad \gamma(x) = \partial D(x),$$

where $r > 0$ is an arbitrary fixed constant. Namely, $D(x)$ is the closed disk of radius r with the center at x, and $\gamma(x)$ is the boundary (circle). We denote by $b(x) \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the flux through $D(x)$ modulo 2π, i.e.,

$$b(x) \int_{D(x)} B(y) \mathrm{d}y_1 \mathrm{d}y_2 \pmod{2\pi\mathbb{Z}}.$$

We identify $\mathbb{T}$ with $[-\pi, \pi)$ and define

$$W(x) = \frac{1}{4\pi^3 r^3} \int_{\gamma(x)} |b(y)|^2 \mathrm{d}y.$$

Then we have

Theorem 2. *$H \geq W$ in the operator sense.*

Note that $0 \leq W(x) \leq (2r^2)^{-1}$ since $|b(x)| \leq \pi$. The advantage of this estimate compared to the AHS estimate [1] is that the right hand side is always nonnegative. On the other hand, $W(x)$ is bounded even if $B(x)$ is not, and the estimate is far from optimal in general. We refer [8] for the general formulation and the proof. We combine this estimate with the argument of Kirsch and Martinelli [6] to prove Theorem 1 (see also [3,5]). In the proof, we need an *a priori* bound on the number of eigenvalues. In [8], we proved this estimate using the uniform boundedness of the magnetic field. We give a different proof of this lemma based on the diamagnetic inequality for magnetic Schrödinger operator with the Neumann boundary condition.

2. Diamagnetic inequality and an *a priori* estimate on the number of eigenvalues

Let $H = (p - A(x))^2$ be a magnetic Schrödinger operator with a vector potential $A \in C^1(\mathbb{R}^d)$, and let H_{Λ_L} be the Hamiltonian on $L^2(\Lambda_L)$ with the Neumann boundary condition. In [8], the boundedness assumption of $B = \mathrm{d}A$ is needed only for the following lemma (Lemma 4 of [8]), which is proved without the assumption here.

Lemma 3. *There exists a constant $C > 0$ depending only on d such that*

$$\frac{1}{|\Lambda_L|} \#\{\text{eigenvalues of } H_{\Lambda_L} \leq E\} \leq C(1 + E^{d/2})$$

for any $E \geq 0$ and for any integer $L > 0$.

At first we prove a diamagnetic inequality for magnetic Schrödinger operators with the Neumann boundary condition.

Lemma 4. *Let $P = H_{\Lambda_L}$, and let $P_0 = p^2$ be the free Hamiltonian on $L^2(\Lambda_L)$ with the Neumann boundary condition. Then*

$$\left|\mathrm{e}^{-tP}\varphi(x)\right| \leq \mathrm{e}^{-tP_0}|\varphi|(x) \qquad for\ t > 0,\ x \in \Lambda_L,$$

where $\varphi \in L^2(\Lambda_L)$. In particular, the integral kernels of e^{-tP} and e^{-tP_0} satisfy

$$\left|\mathrm{e}^{-tP}(x, y)\right| \leq \mathrm{e}^{-tP_0}(x, y) \qquad for\ x, y \in \Lambda_L.$$

Proof. We mimic the proof of Theorem 1.13 of [2]. By a generalized Trotter's formula, we have

$$\mathrm{e}^{-tP} = \operatorname*{s\text{-}lim}_{n\to\infty} \left[\mathrm{e}^{-(t/n)P_1}\mathrm{e}^{-(t/n)P_2} \cdots \mathrm{e}^{-(t/n)P_d}\right]^n, \tag{T}$$

where

$$P_j = (p_j - A_j(x))^2, \qquad j = 1, \dots, d,$$

with the Neumann boundary condition, i.e., $\varphi \in \mathfrak{D}(P_j)$ if and only if $P_j\varphi \in L^2(\Lambda_L)$ (in the distribution sense) and $(p_j - A_j(x))\varphi(x) = 0$ at $x_j = -L$ and L. Let

$$G_j(x) = \int_{-L}^{x_j} A_j(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_d)\mathrm{d}y$$

be a gauge transform such that

$$(p_j - A_j) = \mathrm{e}^{iG_j} p_j \mathrm{e}^{-iG_j}$$

in the distribution sense. Note that if $\varphi \in \mathfrak{D}\,(P_j)$ then $\psi = \mathrm{e}^{-iG_j}\varphi$ satisfies $p_j\psi(x) = 0$ at $x_j = -L$ and L. Hence $P_{j,0} \equiv \mathrm{e}^{iG_j} P_j \mathrm{e}^{-iG_j}$ is p_j^2 with the Neumann boundary condition at the edges. Therefore

$$\mathrm{e}^{-(t/n)P_j} = \mathrm{e}^{iG_j}\mathrm{e}^{-(t/n)P_{j,0}}\mathrm{e}^{-iG_j}$$

and hence

$$\left|\mathrm{e}^{-(t/n)P_j}\varphi(x)\right| \le \mathrm{e}^{-(t/n)P_{j,0}}|\varphi|(x).$$

This implies

$$\left|\left[\mathrm{e}^{-(t/n)P_1}\mathrm{e}^{-(t/n)P_2}\cdots\mathrm{e}^{-(t/n)P_d}\right]^n\varphi(x)\right|$$
$$\le \left[\mathrm{e}^{-(t/n)P_{1,0}}\mathrm{e}^{-(t/n)P_{2,0}}\cdots\mathrm{e}^{-(t/n)P_{d,0}}\right]^n|\varphi|(x) = \mathrm{e}^{-tP_0}|\varphi|(x)$$

and the claim follows from this and (T). □

Proof of Lemma 3. By Lemma 4, we have

$$\mathrm{Tr}\,(\mathrm{e}^{-tP}) \le \mathrm{Tr}(\mathrm{e}^{-tP_0}), \qquad t > 0.$$

Since the eigenvalues of P_0 is explicitly known, we can easily see

$$\mathrm{Tr}\,(\mathrm{e}^{-tP_0}) = \left[\sum_{n=0}^{\infty}\mathrm{e}^{-t(n\pi/L)^2}\right]^d \le \left[1 + \frac{L}{2\sqrt{\pi t}}\right]^d.$$

We set $t = E^{-1}$ to obtain

$$\#\{\text{eigenvalues of } P \le E\} \le e\,\mathrm{Tr}(\mathrm{e}^{-P/E}) \le e\,\mathrm{Tr}\,(\mathrm{e}^{-P_0/E})$$
$$\le e\left(1 + \frac{L\sqrt{E}}{2\sqrt{\pi}}\right)^d \le CL^d(1 + E^{d/2})$$

with some $C > 0$ if $L \ge 1$. □

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