Proc. Indian Acad. Sci. (Math. Sci.) Vol. 112, No. 1, February 2002, pp. 107–116.

On perturbation of eigenvalues embedded at thresholds in a two channel model

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Abstract. We present some results on the perturbation of eigenvalues embedded at thresholds in a two channel model Hamiltonian with a small off-diagonal perturbation. Examples are given of the various types of behavior of the eigenvalue under perturbation.

Keywords. Schrödinger operator; perturbation theory; threshold eigenvalue; two channel model.

1. Introduction

In this paper we consider two channel Hamiltonians of the form $H(g) = H_0 + gV$, where

$$H_0 = \begin{pmatrix} H_a & 0 \\ 0 & H_b \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & V_{ab} \\ V_{ba} & 0 \end{pmatrix} \tag{1.1}$$

on a Hilbert space $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_b$. Here H_a and H_b are self-adjoint operators on $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively. We assume $V_{ab} \in \mathcal{B}(\mathcal{H}_b, \mathcal{H}_a)$, the bounded operators, and $V_{ba} = V_{ab}^*$. The direct sum structure of H_0 means that one can easily find examples with eigenvalues embedded at thresholds.

To explain the type of results obtained, let us assume that the operator H_a is a Schrödinger operator with absolutely continuous spectrum $[0, \infty)$ and discrete spectrum in $(-\infty, 0)$. We also assume that the threshold 0 is a regular point, in the sense that there exists a Hilbert space $\mathcal{K}_a$, densely and continuously embedded in $\mathcal{H}_a$, such that we have an asymptotic expansion of the resolvent $R_a(\zeta) = (H_a - \zeta)^{-1}$ of the form

$$R_a(\zeta) = G_0 + i\zeta^{1/2}G_1 - \zeta G_2 - i\zeta^{3/2}G_3 + 0(\zeta^2) \tag{1.2}$$

as $\zeta \to 0$, $\operatorname{Im}\zeta > 0$. The expansion is assumed to hold in the norm topology of $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Assume furthermore that 0 is a simple isolated eigenvalue of H_b. Then H_0 has an eigenvalue embedded at the threshold 0. We are interested in understanding what happens to this eigenvalue under small off-diagonal perturbations. A number of results on this problem (and several other related problems) were obtained in [7]. In this paper we obtain some further detailed results. We also give a number of examples concerning the assumptions in our main result, Theorem 2.4.

The results obtained here state that certain intervals around the threshold are free of eigenvalues. Both end points may depend on the coupling constant g. The precise behavior depends on the interaction, and its relation to the eigenfunction of H_b, the reduced resolvent of H_b, and the coefficients in the expansion (1.2). We refer to the statements in Theorem 2.4.

Let us now comment on the literature. As mentioned above, the results presented here extend to those in [7]. We use the technique of that paper here, which consists in combining the asymptotic expansion with the Feshbach formula for the resolvent of $H(g)$ and a certain factorization technique.

A general result on absorption of eigenvalues in the continuum was obtained in [14]. Our results are related to the results on coupling constant thresholds. For ordinary Schrödinger operators with sufficiently rapidly decaying potentials a number of such results are given in [10, 11, 5]. Some results have also been obtained on perturbations of Schrödinger operators with periodic potentials [12, 2], and for the Dirac operator [9].

The difference between results on coupling constant thresholds and our results is that we give intervals both below (in the resolvent set of $H(g)$) and above (in the absolutely continuous spectrum) the threshold which contain no eigenvalues. The papers on coupling constant thresholds only consider the behavior below the threshold. On the other hand, these papers give results on the existence of eigenvalues, and on their dependence on the coupling constant in the form of asymptotic expansions. It is possible to extend some of our results to the existence of eigenvalues below the threshold. For example, this is the case for Theorem 2.4(i), see Remark 2.6.

Under additional assumptions one can obtain a meromorphic continuation of the resolvent (in the variable $\zeta^{1/2}$) around a neighborhood of the threshold 0, and then give a unified discussion of eigenvalues and resonances. There are several results of this type in the literature, see for example [13, 3, 4]. In the two channel model considered here a number of problems concerning resonances remain unresolved. We hope to return to these problems elsewhere.

2. The Theorem

We use the notation $R_a(\zeta) = (H_a - \zeta)^{-1}$, etc. for the resolvents, and we introduce the operator

$$T_b(\zeta) = H_b - \zeta - g^2 V_{ba} R_a(\zeta) V_{ab}. \tag{2.1}$$

The Feshbach formula is the representation of $R(g;\zeta) = (H(g) - \zeta)^{-1}$, for $\operatorname{Im}\zeta \neq 0$, given by

$$R(g;\zeta) = \begin{pmatrix} R_a(\zeta) + g^2 R_a(\zeta) V_{ab} T_b(\zeta)^{-1} V_{ba} R_a(\zeta) & -g R_a(\zeta) V_{ab} T_b(\zeta)^{-1} \\ -g T_b(\zeta)^{-1} V_{ba} R_a(\zeta) & T_b(\zeta)^{-1} \end{pmatrix} \tag{2.2}$$

Let us now introduce our assumptions. To simplify the presentation we have assumed that the threshold is located at energy 0. A change of variables yields the general result.

Assumption 2.1. Assume that 0 is a simple isolated eigenvalue of H_b, with normalized eigenfunction ψ_b.

We denote the eigenprojection by $P_b = \langle \psi_b, \cdot \rangle \psi_b$. The reduced resolvent is given by

$$C_b = \lim_{\zeta \to 0} (I_b - P_b) R_b(\zeta), \tag{2.3}$$

and we have the norm convergent expansion

$$R_b(\zeta) = -\frac{1}{\zeta} P_b + \tilde{R}_b(\zeta) = -\frac{1}{\zeta} P_b + \sum_{n=0}^{\infty} \zeta^n C_b^{n+1}. \tag{2.4}$$

The expansion is valid for $0 < |\zeta| < \delta$ for some small $\delta > 0$.

We need asymptotic expansion assumptions on $R_a(\zeta)$ at the threshold. The following assumption is modeled on the known expansions for a Schrödinger operator in an odd dimensional space, see [6]. Results can also be given and modeled on the even dimensional spaces.

Assumption 2.2. Assume that $0 \in \sigma(H_a)$. Assume that there exists a Hilbert space $\mathcal{K}_a$, densely and continuously embedded in $\mathcal{H}_a$, and operators $G_j \in \mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$, with $G_j^* = G_j$, for $j = 0, 1, 2, 3$, such that

$$R_a(\zeta) = G_0 + i\zeta^{1/2} G_1 - \zeta G_2 - i\zeta^{3/2} G_3 + \zeta^2 \rho_a(\zeta) \tag{2.5}$$

as $\zeta \to 0$, $\operatorname{Im} \zeta > 0$, in the norm topology of $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Here $\rho_a(\zeta)$ is a bounded function with values in $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$.

We use the following assumption on the interaction.

Assumption 2.3. We assume that $V_{ab} \in \mathcal{B}(\mathcal{H}_b, \mathcal{K}_a)$, for the space $\mathcal{K}_a$ from Assumption 2.2.

Under these assumptions we can introduce four constants that will be needed in the statement of the theorem.

$$\alpha_0 = \langle \psi_b, V_{ba} G_0 V_{ab} \psi_b \rangle, \tag{2.6}$$

$$\beta_0 = \langle \psi_b, V_{ba} G_1 V_{ab} \psi_b \rangle, \tag{2.7}$$

$$\gamma_0 = \langle \psi_b, V_{ba} G_0 V_{ab} C_b V_{ba} G_0 V_{ab} \psi_b \rangle, \tag{2.8}$$

$$\omega_0 = 2 \operatorname{Re} \langle \psi_b, V_{ba} G_0 V_{ab} C_b V_{ba} G_1 V_{ab} \psi_b \rangle. \tag{2.9}$$

The notation $\delta(g) \asymp g^{2n}$ means that there exist $\eta_0 > 0$, $c_1 > 0$, and $c_2 > 0$, such that for all g with $|g| < \eta_0$ we have $c_1 g^{2n} \leq \delta(g) \leq c_2 g^{2n}$.

Note that parts (i)–(iv) of the following theorem are re-statements of results in [7]. The results actually hold under slightly weaker assumptions on the asymptotic expansion than imposed here.

Theorem 2.4. *Let Assumptions* 2.1, 2.2, *and* 2.3 *hold. Then we have the following results.*

(i) *Assume that $\alpha_0 > 0$. Then there exist $\eta_0 > 0$, $\delta_0 > 0$, and $\delta_l(g) \asymp g^2$ such that $(-\delta_l(g), \delta_0) \cap \sigma_{pp}(H(g)) = \emptyset$ for all g with $0 < |g| < \eta_0$.*

(ii) *Assume that $\alpha_0 < 0$ and $\beta_0 \neq 0$. Then there exist $\eta_0 > 0$ and $\delta_0 > 0$ such that $(-\delta_0, \delta_0) \cap \sigma_{pp}(H(g)) = \emptyset$ for all g with $0 < |g| < \eta_0$.*

(iii) *Assume that $\alpha_0 < 0$ and $\beta_0 = 0$. Then there exist $\eta_0 > 0$, $\delta_0 > 0$, and $\delta_r(g) \asymp g^2$ such that $(-\delta_0, \delta_r(g)) \cap \sigma_{pp}(H(g)) = \emptyset$ for all g with $0 < |g| < \eta_0$.*

(iv) *Assume that $\alpha_0 = 0$, $\beta_0 \neq 0$, and $\gamma_0 \neq 0$. Then there exist $\eta_0 > 0$, $\delta_0 > 0$, and $\delta_l(g) \asymp g^4$ such that $(-\delta_l(g), \delta_0) \cap \sigma_{pp}(H(g)) = \emptyset$ for all g with $0 < |g| < \eta_0$.*

(v) *Assume that* $\alpha_0 = 0$, $\beta_0 = 0$, *and* $\gamma_0 > 0$. *Then there exist* $\eta_0 > 0$, $\delta_0 > 0$, *and* $\delta_l(g) \asymp g^8$ *such that* $(-\delta_l(g), \delta_0) \cap \sigma_{pp}(H(g)) = \emptyset$ *for all* g *with* $0 < |g| < \eta_0$.

(vi) *Assume that* $\alpha_0 = 0$, $\beta_0 = 0$, $\gamma_0 < 0$, *and* $\omega_0 \neq 0$. *Then there exist* $\eta_0 > 0$ *and* $\delta_0 > 0$, *such that* $(-\delta_0, \delta_0) \cap \sigma_{pp}(H(g)) = \emptyset$ *for all* g *with* $0 < |g| < \eta_0$.

(vii) *Assume that* $\alpha_0 = 0$, $\beta_0 = 0$, $\gamma_0 < 0$, *and* $\omega_0 = 0$. *Then there exist* $\eta_0 > 0$, $\delta_0 > 0$, *and* $\delta_r(g) \asymp g^8$ *such that* $(-\delta_0, \delta_r(g)) \cap \sigma_{pp}(H(g)) = \emptyset$ *for all* g *with* $0 < |g| < \eta_0$.

Proof. Parts (i)–(iv) were proved in [7]. We will outline here the proof of parts (v)–(vii). Thus we assume $\alpha_0 = 0$ and $\beta_0 = 0$.

The idea of the proof is to show that the operator $T_b(\zeta)$, defined in (2.1), is invertible with a uniformly bounded inverse in a neighborhood of 0, which may depend on g. We start by factoring this operator as follows:

$$T_b(\zeta) = (I_b - g^2 V_{ba} R_a(\zeta) V_{ab} R_b(\zeta))(H_b - \zeta) = \Phi(\zeta)(H_b - \zeta). \tag{2.10}$$

We write (2.4) as

$$R_b(\zeta) = -\frac{1}{\zeta} P_b + C_b + \zeta \rho_b(\zeta),$$

where $\rho_b(\zeta) = O(1)$ in $\mathcal{B}(\mathcal{H}_b)$ as $\zeta \to 0$. We now insert this expansion and the expansion from (2.5), and reorder the terms to get

$$\begin{aligned}\Phi(\zeta) = I_b &+ \frac{g^2}{\zeta} V_{ba} G_0 V_{ab} P_b + \frac{i g^2}{\zeta^{1/2}} V_{ba} G_1 V_{ab} P_b \\ &- g^2 (V_{ba} G_0 V_{ab} C_b + V_{ba} G_2 V_{ab} P_b) \\ &- i g^2 \zeta^{1/2} (V_{ba} G_1 V_{ab} C_b + V_{ba} G_3 V_{ab} P_b) \\ &+ g^2 \zeta (V_{ba} G_0 V_{ab} \rho_b(\zeta) - V_{ba} \rho_a(\zeta) V_{ab} P_b - V_{ba} G_2 V_{ab} C_b).\end{aligned}$$

We collect the singular terms in the operator $S(\zeta)$, defined by

$$S(\zeta) = I_b + \frac{g^2}{\zeta} (V_{ba} G_0 V_{ab} P_b + i \zeta^{1/2} V_{ba} G_1 V_{ab} P_b). \tag{2.11}$$

The remaining non-singular terms are collected in the operator $K(\zeta)$, leading to a decomposition $\Phi(\zeta) = S(\zeta) + K(\zeta)$. Due to the assumptions $\alpha_0 = 0$ and $\beta_0 = 0$, a direct computation shows that for $\zeta \neq 0$ the operator $S(\zeta)$ is invertible, and its inverse is given by

$$S(\zeta)^{-1} = I_b - \frac{g^2}{\zeta} (V_{ba} G_0 V_{ab} P_b + i \zeta^{1/2} V_{ba} G_1 V_{ab} P_b). \tag{2.12}$$

This means that we can write $\Phi(\zeta) = (I_b + K(\zeta) S(\zeta)^{-1}) S(\zeta)$. We use the notation $U(\zeta) = I_b + K(\zeta) S(\zeta)^{-1}$. Multiplying out and isolating the singular terms we get the expression

$$\begin{aligned}U(\zeta) = I_b &+ \frac{g^4}{\zeta} V_{ba} G_0 V_{ab} C_b V_{ba} G_0 V_{ab} P_b \\ &+ \frac{i g^4}{\zeta^{1/2}} (V_{ba} G_0 V_{ab} C_b V_{ba} G_1 V_{ab} P_b + V_{ba} G_1 V_{ab} C_b V_{ba} G_0 V_{ab} P_b) \\ &+ g^2 O(1).\end{aligned} \tag{2.13}$$

We denote the singular terms plus the identity in (2.13) by $S_1(\zeta)$. A simple direct computation shows that this operator is invertible, if $\zeta \neq 0$ and $\zeta + i\zeta^{1/2}g^4\omega_0 + g^4\gamma_0 \neq 0$, and that the inverse is given by

$$\begin{aligned} S_1(\zeta)^{-1} = I_b &- \frac{g^4}{\zeta + i\zeta^{1/2}g^4\omega_0 + g^4\gamma_0} V_{ba}G_0V_{ab}C_bV_{ba}G_0V_{ab}P_b \\ &- \frac{i\zeta^{1/2}g^4}{\zeta + i\zeta^{1/2}g^4\omega_0 + g^4\gamma_0}(V_{ba}G_0V_{ab}C_bV_{ba}G_1V_{ab}P_b \\ &+ V_{ba}G_1V_{ab}C_bV_{ba}G_0V_{ab}P_b). \end{aligned} \tag{2.14}$$

Thus we can write

$$U(\zeta) = S_1(\zeta) + L(\zeta) = (I_b + L(\zeta)S_1(\zeta)^{-1})S_1(\zeta) = W(\zeta)S_1(\zeta).$$

For each ζ satisfying $\zeta + i\zeta^{1/2}g^4\omega_0 + g^4\gamma_0 \neq 0$ the operator $W(\zeta)$ is invertible, for $|g|$ sufficiently small. Thus we have a factorization of $T_b(\zeta) = W(\zeta)S_1(\zeta)S(\zeta)(H_b - \zeta)$, where each term is invertible. In the inverse

$$T_b(\zeta)^{-1} = R_b(\zeta)S(\zeta)^{-1}S_1(\zeta)^{-1}W(\zeta)^{-1},$$

the last two factors are regular at $\zeta = 0$, whereas the first two factors have a singularity at $\zeta = 0$. Now it turns out that the singularities cancel, when we multiply out the first three factors. We start with the first two factors, using the notation in (2.4)

$$\begin{aligned} R_b(\zeta)S(\zeta)^{-1} &= \left(-\frac{1}{\zeta}P_b + \tilde{R}_b(\zeta)\right) \\ &\quad \times \left(I_b - \frac{g^2}{\zeta}\left(V_{ba}G_0V_{ab} + i\zeta^{1/2}V_{ba}G_1V_{ab}\right)P_b\right) \\ &= \tilde{R}_b(\zeta) - \frac{1}{\zeta}(I_b + (\cdots))P_b. \end{aligned} \tag{2.15}$$

In this computation we have used the assumptions to get $P_bV_{ba}G_0V_{ab}P_b = \alpha_0 P_b = 0$ and $P_bV_{ba}G_1V_{ab}P_b = \beta_0P_b = 0$. Thus the coefficients to the $1/\zeta^2$-term and the $1/\zeta^{3/2}$-term are both zero. Now in $R_b(\zeta)S(\zeta)^{-1}S_1(\zeta)^{-1}$ we need to compute

$$\begin{aligned} P_bS_1(\zeta)^{-1} &= P_b - \frac{g^4}{\zeta + i\zeta^{1/2}g^4\omega_0 + g^4\gamma_0} P_bV_{ba}G_0V_{ab}C_bV_{ba}G_0V_{ab}P_b \\ &\quad - \frac{i\zeta^{1/2}g^4}{\zeta + i\zeta^{1/2}g^4\omega_0 + g^4\gamma_0}(P_bV_{ba}G_0V_{ab}C_bV_{ba}G_1V_{ab}P_b \\ &\quad + P_bV_{ba}G_1V_{ab}C_bV_{ba}G_0V_{ab}P_b) \\ &= P_b - \frac{g^4}{\zeta + i\zeta^{1/2}g^4\omega_0 + g^4\gamma_0}\gamma_0P_b - \frac{i\zeta^{1/2}g^4}{\zeta + i\zeta^{1/2}g^4\omega_0 + g^4\gamma_0}\omega_0P_b \\ &= \frac{\zeta}{\zeta + i\zeta^{1/2}g^4\omega_0 + g^4\gamma_0}P_b. \end{aligned}$$

Thus the $1/\zeta$ singularity in (2.15) is canceled by the ζ factor in the numerator above.

It remains to analyse the zeroes of the polynomial $z^2 + izg^4\omega_0 + g^4\gamma_0$. This analysis leads to the three cases (v)–(vii) in the theorem. The straightforward details are omitted. □

Remark 2.5. Case (v) of the theorem can be generalized to arbitrary finite rank of P_b. The assumption $\alpha_0 = 0$ is then replaced by the assumption that the operator $P_b V_{ba} G_0 V_{ab} P_b = 0$, and analogously for β_0. The assumption $\gamma_0 > 0$ is replaced by the assumption that $P_b V_{ba} G_0 V_{ab} C_b V_{ba} G_0 V_{ab} P_b$ is strictly positive and invertible in $\mathcal{B}(P_b \mathcal{H}_b)$. See also the result ([7] Theorem 3.9), giving such a generalization of case (i) in the theorem. The cases (ii)–(iv), (vi), and (vii) can also be generalized, but the statement of the results is rather complicated.

Remark 2.6. Let us note that in some cases the results in the theorem can be supplemented with results showing the existence of eigenvalues near zero, with behavior $-\delta_l(g)$ as $g \to 0$. Such results can be obtained by the application of variational arguments to the operator $H_b - g^2 V_{ba} G_0 V_{ab}$, followed by a perturbation argument. This argument can be applied to case (i) of the theorem.

Remark 2.7. Note that our proof shows that the limiting absorption principle holds for $H(g)$ on the right hand intervals $(0, \delta_0)$ and $(0, \delta_r(g))$, such that the spectrum of $H(g)$ is absolutely continuous on these intervals (or these intervals are in the resolvent set, since we have not excluded the case $G_1 = G_3 = 0$).

3. Examples

In this section we give some examples showing that all cases in Theorem 2.4 may occur. For the cases (i)–(iii) we can give fairly simple examples, which are natural in the two channel context. For the cases (iv)–(vii) we give examples in a simplified framework. Note how the examples relate spectral information on H_a and H_b to the conditions on the constants (2.6)–(2.9).

3.1 *The cases (i)–(iii)*

We start by defining the spaces and the operators. Let

$$H_a = -\Delta + V(x) \text{ on } \mathcal{H}_a = L^2(\mathbf{R}^3), \tag{3.1}$$

where we assume that $V(x)$ is a real-valued function satisfying the decay condition

$$|V(x)| \le C(1 + |x|)^{-9-\varepsilon} \tag{3.2}$$

for some $\varepsilon > 0$. We also assume that H_a has at least one negative eigenvalue.

Let $L^{2,s}(\mathbf{R}^3) = L^2(\mathbf{R}^3, (1+|x|)^{2s}\,\mathrm{d}x)$, $s \in \mathbf{R}$, denote the weighted spaces. We assume that 0 is a regular point for H_a, which means that the equation $(-\Delta + V(x))\psi = 0$ has no nonzero solution in $L^{2,s}(\mathbf{R}^3)$ for $-3/2 \le s < -1/2$, see [6].

We take $\mathcal{K}_a = L^{2,s_0}(\mathbf{R}^3)$ for a fixed $s_0 > 9/2$. It then follows from our assumptions and the results in [6] that we have an asymptotic expansion

$$R_a(\zeta) = G_0 + i\zeta^{1/2} G_1 - \zeta G_2 - i\zeta^{3/2} G_3 + \zeta^2 \rho_a(\zeta) \tag{3.3}$$

as $\zeta \to 0$, $\operatorname{Im} \zeta > 0$, where the expansion holds in the norm topology of $\mathcal{B}(\mathcal{K}_a, \mathcal{K}_a^*)$. Thus we have verified the conditions in Assumption 2.2.

For operator H_b we can take any self-adjoint operator on a Hilbert space $\mathcal{H}_b$ such that 0 is an isolated simple eigenvalue for H_b. The normalized eigenfunction is denoted by ψ_b. Thus Assumption 2.1 holds.

Let $\xi_a \in \mathcal{K}_a$ and define

$$V_{ab} = \langle \psi_b, \cdot \rangle \xi_a, \quad V_{ba} = \langle \xi_a, \cdot \rangle \psi_b.$$

Here we have used the notation $\langle \cdot\,, \cdot \rangle$ both for the inner products and for the duality between $\mathcal{K}_a$ and $\mathcal{K}_a^*$. For this interaction Assumption 2.3 clearly holds.

A simple computation shows that

$$\alpha_0 = \langle \xi_a,\ G_0 \xi_a \rangle, \quad \beta_0 = \langle \xi_a,\ G_1 \xi_a \rangle.$$

To give the examples we need some preparation. We first recall that for $f \in C_0^\infty(\mathbf{R}\backslash\{0\})$ we have $f(H_a) \in \mathcal{B}(\mathcal{K}_a)$, see for example [8] and the references therein. Let $\eta_a \in \mathcal{K}_a$ and $\xi_a = f(H_a)\eta_a$, and let $E_a(\lambda)$ denote the spectral measure of H_a. Then we have

$$\langle \xi_a,\ G_0 \xi_a \rangle = \lim_{\substack{\zeta \to 0 \\ \mathrm{Re}\zeta < 0, \mathrm{Im}\zeta = 0}} \langle \xi_a, R_a(\zeta) \xi_a \rangle = \int_{-\infty}^{\infty} \frac{|f(\lambda)|^2}{\lambda} \mathrm{d}\langle \eta_a,\ E_a(\lambda)\eta_a \rangle. \tag{3.4}$$

This result shows that we can find $f \in C_0^\infty((0,\infty))$ and $\eta_a \in \mathcal{K}_a$, such that with $\xi_a = f(H_a)\eta_a$ we have $\alpha_0 = \langle \xi_a,\ G_0 \xi_a \rangle > 0$.

To get information on β_0 we need to use the explicit form of G_1 from [6] (see eq. (6.3)). We have

$$G_1 = \frac{1}{4\pi} \langle (1 + K_0 V)^{-1} 1, \cdot \rangle (1 + K_0 V)^{-1} 1, \tag{3.5}$$

where K_0 denotes the operator given by the integral kernel $1/(4\pi|x - y|)$. We have assumed that H_a has at least one negative eigenvalue. Let ϕ_a be one of the normalized eigenfunctions. We know that $\phi_a \in \mathcal{K}_a$, since such eigenfunctions decay exponentially. There are two possibilities to consider. Assume first that

$$\langle (1 + K_0 V)^{-1} 1,\ \phi_a \rangle \neq 0.$$

Then we can take $\xi_a = \phi_a$. It follows from (3.4) that $\alpha_0 < 0$, and the above condition implies $\beta_0 \neq 0$.

In the second case we assume

$$\langle (1 + K_0 V)^{-1} 1,\ \phi_a \rangle = 0 \tag{3.6}$$

for all eigenfunctions ϕ_a corresponding to negative eigenvalues. Taking $\xi_a = \phi_a$ we have $\alpha_0 < 0$ and $\beta_0 = 0$. To get an example with $\alpha_0 < 0$ and $\beta_0 \neq 0$ we use that (3.6) implies the existence of $\eta_a \in \mathcal{K}_a$ and $f \in C_0^\infty((0,\infty))$ such that

$$\langle (1 + K_0 V)^{-1} 1,\ f(H_a)\eta_a \rangle \neq 0. \tag{3.7}$$

We then choose an appropriate value for c in $\xi_a = \phi_a + c f(H_a)\eta_a$, in order to get $\alpha_0 < 0$. Due to (3.6) and (3.7) we also have $\beta_0 \neq 0$.

Thus we have shown that the three cases (i), (ii), and (iii) can be realized in this example. Note that the examples cover the confining channel model, see [1] and ([7], §5.2).

3.2 *The cases (iv)–(vii)*

To cover these cases we assume that H_a is a two channel Hamiltonian of a particular form. Note that we now change the definitions of $\mathcal{H}_a$ and H_a. We take

$$\mathcal{H}_a = L^2(\mathbf{R}^3) \oplus \mathbf{C}$$

and use an obvious matrix notation for operators on this space. Let

$$H_a = \begin{pmatrix} -\Delta + V(x) & 0 \\ 0 & T \end{pmatrix}. \tag{3.8}$$

Here V is assumed to satisfy (3.2). As above we also assume that $-\Delta + V(x)$ has at least one negative eigenvalue. Let $\mu_0 \in \mathbf{R}\backslash\{0\}$. Then we define $Tz = \mu_0 z$ on $\mathbf{C}$. As an operator on $\mathbf{C}$ its resolvent has the expansion

$$(T - \zeta)^{-1} = \frac{1}{\mu_0} + \frac{1}{\mu_0^2}\zeta + \frac{1}{\mu_0^3}\zeta^2 + O(\zeta^3)$$

as $\zeta \to 0$. Combining this result with the results from subsection 3.1 we see that the new operator H_a defined in (3.8) satisfies Assumption 2.2 with $\mathcal{K}_a = L^{2,s_0}(\mathbf{R}^3) \oplus \mathbf{C}$, $s_0 > 9/2$. The first coefficient in the expansion is

$$\begin{pmatrix} G_0 & 0 \\ 0 & \frac{1}{\mu_0} \end{pmatrix},$$

where we continue to use the notation from subsection 3.1 for the expansion of the first component in the new H_a. The other coefficients have similar representations.

For operator H_b we take a self-adjoint operator on $\mathcal{H}_b$ with 0 as a simple isolated eigenvalue. The normalized eigenfunction is again denoted by ψ_b. Let E_b denote the spectral measure of H_b. We also assume that $E_b((-\infty, 0)) \neq 0$ and $E_b((0, \infty)) \neq 0$.

To define the interaction we take $\xi_a \in L^{2,s_0}(\mathbf{R}^3)$ and $\xi_b \in \mathcal{H}_b$. The choice of ξ_a and ξ_b will be specified later, depending on the case under consideration. For $\phi_b \in \mathcal{H}_b$, we define

$$V_{ab}\phi_b = \begin{pmatrix} \langle \xi_b, \phi_b \rangle \xi_a \\ \langle \psi_b, \phi_b \rangle 1 \end{pmatrix}.$$

The adjoint is then given for $(\eta_a, z) \in \mathcal{H}_a$ by

$$V_{ba}\begin{pmatrix} \eta_a \\ z \end{pmatrix} = \langle \xi_a, \eta_a \rangle + z\psi_b.$$

With these definitions it is clear that Assumption 2.3 is also satisfied.

A simple computation yields the following expressions for the constants in (2.6)–(2.9).

$$\begin{aligned}
\alpha_0 &= |\langle \xi_b, \psi_b \rangle|^2 \langle \xi_a, G_0 \xi_a \rangle + \frac{1}{\mu_0}, \\
\beta_0 &= |\langle \xi_b, \psi_b \rangle|^2 \langle \xi_a, G_1 \xi_a \rangle, \\
\gamma_0 &= |\langle \xi_b, \psi_b \rangle|^2 |\langle \xi_a, G_0 \xi_a \rangle|^2 \langle \xi_b, C_b \xi_b \rangle, \\
\omega_0 &= 2|\langle \xi_b, \psi_b \rangle|^2 |\langle \xi_a, G_0 \xi_a \rangle|^2 \langle \xi_a, G_1 \xi_a \rangle \langle \xi_b, C_b \xi_b \rangle.
\end{aligned}$$

Let us now go through the various cases. We start with the case (iv). Using the results from subsection 3.1 we can find ξ_a such that $\langle \xi_a, \ G_j \xi_a \rangle \neq 0$, $j = 0, 1$. Our assumptions on the spectrum of H_b imply that the reduced resolvent $C_b \neq 0$. Thus we can find η_b such that $\langle \eta_b, \ C_b \eta_b \rangle \neq 0$ and $\langle \psi_b, \ \eta_b \rangle = 0$. We then take $\xi_b = \psi_b + \eta_b$. Finally we let $\mu_0 = -1/\langle \xi_a, \ G_0 \xi_a \rangle$. It follows that we have $\alpha_0 = 0$, $\beta_0 \neq 0$, and $\gamma_0 \neq 0$.

For the case (v) we again use results from subsection 3.1. It follows from those results that we can find ξ_a, such that $\langle \xi_a, \ G_0 \xi_a \rangle \neq 0$, and $\langle \xi_a, \ G_1 \xi_a \rangle = 0$. This time we take $\xi_b = \psi_b + \eta_b$, with $\eta_b \in E_b((0, \infty))\mathcal{H}_b$, $\eta_b \neq 0$. Then we get $\langle \eta_b, \ C_b \eta_b \rangle > 0$. We also take $\mu_0 = -1/\langle \xi_a, \ G_0 \xi_a \rangle$. It follows that we have obtained an example with $\alpha_0 = 0$, $\beta_0 = 0$, and $\gamma_0 > 0$.

The case (vii) is very similar to the case (v). Note that $\beta_0 = 0$ implies $\omega_0 = 0$. Thus we only need to modify the above construction by taking $\eta_b \in E_b((-\infty, 0))\mathcal{H}_b$, $\eta_b \neq 0$.

The case (vi) requires a different construction, since $\beta_0 = 0$ implies $\omega_0 = 0$. If we take as T a rank 2 operator, the above construction is easily modified to accommodate this case. We omit the details.

Acknowledgment

I want to thank the organizers of the Goa workshop on ‘Spectral and Inverse Spectral Theories of Schrödinger Operators’ for organizing a most interesting workshop. The author is associated with Centre for Mathematical Physics and Stochastics, funded by the Danish National Research Foundation.

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