Proc. Indian Acad. Sci. (Math. Sci.) Vol. 112, No. 1, February 2002, pp. 147–162.

Lifshitz tails for random perturbations of periodic Schrödinger operators

FRÉDÉRIC KLOPP

Département de Mathématique, Institut Galilée, U.M.R. 7539 C.N.R.S,
Université de Paris-Nord, 99 Avenue J.-B. Clément, F-93430 Villetaneuse, France
E-mail: klopp@math.univ-paris13.fr

Abstract. The present paper is a non-exhaustive review of Lifshitz tails for random perturbations of periodic Schrödinger operators. It is not our goal to review the whole literature on Lifshitz tails; we will concentrate on a single model, the continuous Anderson model.

Keywords. Random Schrödinger operators; density of states; Lifshitz tails.

0. Introduction

In the early 60's (see [24,26]), I M Lifshitz produced a heuristic showing that, at the fluctuational band edges of the spectrum of a random Schrödinger operator, the density of states decays exponentially fast. This differs dramatically from the behavior of the density of states of a periodic Schrödinger operator: in this case, the band edge decay is polynomial. One of the major consequences of this difference is that the band edge spectral behaviors for these two classes of operators are radically different: in the random case, the spectrum is localized and, in the periodic case, the spectrum is extended. Indeed Lifshitz tails play a crucial role in the proof of band-edge localization (see e.g. [14,2,15,18]).

To be more specific, let us turn to the random model central to our study. To simplify the exposition, we will not give technically optimal assumptions.

1. The continuous Anderson model

Let W be a $\mathbb{Z}^d$-periodic potential in $L^p_{\text{loc}}(\mathbb{R}^d)$ (where $p = 2$ if $d \leq 3$, $p > 2$ if $d = 4$ and $p > d/2$ if $d \geq 5$) and consider the periodic Schrödinger operator $H = -\Delta + W$ acting on $L^2(\mathbb{R}^d)$. It is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$; let Σ_p be its spectrum.

The *continuous Anderson model* is defined to be the random Schrödinger operator

$$H_\omega = H + V_\omega = H + \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma V_\gamma, \tag{1.1}$$

where $V_\gamma(\cdot) = V(\cdot - \gamma)$, $V : \mathbb{R}^d \to \mathbb{R}$ is a potential and $(\omega_\gamma)_{\gamma \in \mathbb{Z}^d}$ are independent identically distributed random variables. We assume that

(H.1):

(1) $0 \not\equiv V$ is continuous and decaying sufficiently fast for the sum in (1.1) to converge.

(2) V is non-negative.
(3) The i.i.d. random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are bounded and their support is the interval $[\omega^-,\omega^+]$.

Let us recall a few well-known facts from the theory of random Schrödinger operators (see e.g. [34,10] or [4]). For all ω, H_ω is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$, and we denote by H_ω its self-adjoint extension. There exists Σ, a closed subset of $\mathbb{R}$ such that, for almost all ω, the spectrum of H_ω coincides with Σ. This set is called the (*almost sure*) *spectrum* of H_ω. These results are valid for ergodic operators much more general than (1.1). Our assumptions on V_ω and H imply that Σ is lower semi-bounded.

It follows from the Bethe–Sommerfeld conjecture (see e.g. [39,9]) and the standard characterization of Σ using admissible periodic operators (see e.g. [11,34]) that Σ is a finite union of intervals (see figure 1). The connected components of $\mathbb{R}\setminus\Sigma$ will be the gaps of Σ. The edges of the gaps are the points in $\Sigma\cap\overline{\mathbb{R}\setminus\Sigma}$.

One defines the integrated density of states of H_ω in the following way. Let $H^D_{\omega,l}$ be the Dirichlet restriction of H_ω to the cube Λ_l centered in 0 of side length l. The operator $H^D_{\omega,l}$ has only discrete spectrum. For $E\in\mathbb{R}$, we define

$$N^D_{\omega,l}(E)=\frac{1}{\mathrm{Vol}(\Lambda_l)}\sharp\{\text{ eigenvalues of } H^D_{\omega,l}\le E\},\tag{1.2}$$

where $\sharp\mathcal{E}$ denotes the number of points of the set $\mathcal{E}$ and $\mathrm{Vol}(\Lambda_l)$, the volume of Λ_l.

Then, ω-almost surely, $N^D_{\omega,l}(E)$ has a limit when $l\to+\infty$. This limit is independent of the realization of ω. It is the *integrated density of states* of H_ω. We will denote it by $N(E)$. Physically, it represents the number of states per unit of volume below energy E for a system governed by H_ω. The function N is non random, non decreasing and right continuous. Its set of increasing points is Σ.

The integrated density of states is the main object of our study. Consider a gap in Σ, say (E_+,E_-) (see figure 2). Typically, if the random perturbation V_ω is not too large, this gap will be obtained by shrinking slightly a gap of Σ_p, say (e_+,e_-).

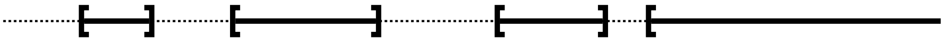

Figure 1. A typical Σ.

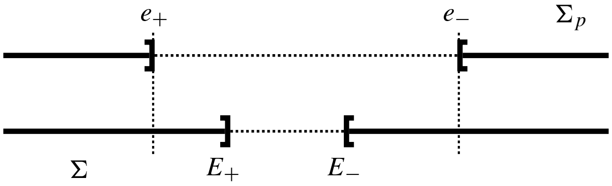

Figure 2. The band edges for Σ and Σ_p.

Our main question then is:

Question. How does $N(E)$ behave near $E_{\pm}$?

The case of E_+ and E_- are similar; so we only state the results in the case of E_- i.e. for lower band edges.

2. More assumptions

Whether answers to the above Question are known or not depends crucially on the band edge one considers, as well as on the decay properties of the single site potential. As far as the band edge is concerned, one can distinguish two cases:

(1) The bottom of the spectrum,

(2) Internal band edges.

As for the decay of V, there are also two cases to be distinguished:

(1) There exists $\nu > d+2$ and $0 \leq g_+$, a continuous non identically vanishing function, such that, for any $\gamma \in \mathbb{Z}^d$ and every $x \in C_0$, one has

$$V(x+\gamma) \cdot (1+ \mid \gamma \mid)^{\nu} \leq g_+(x). \tag{2.1}$$

(2) There exists $\nu \in (d, d+2]$ and $0 \leq g_- \leq g_+$, two continuous non vanishing functions, such that, for any $\gamma \in \mathbb{Z}^d$ and every $x \in C_0$, one has

$$g_-(x) \leq V(x+\gamma) \cdot (1+ \mid \gamma \mid)^{\nu} \leq g_+(x). \tag{2.2}$$

There is one more parameter that may change the behavior of the density of states at band edges: it is the behavior of the random variables at the edges of their support. Therefore, in the sequel we shall assume that the common distribution of the $(\omega_\gamma)_{\gamma \in \mathbb{Z}^d}$ satisfies

$$\lim_{\varepsilon \to 0} \frac{\log|\log \mathbb{P}\{|\omega_0 - \omega^-| \leq \varepsilon\}|}{-\log \varepsilon} = 0. \tag{2.3}$$

Of course, in the case of upper band edges, one has to make the symmetric assumption on the behavior of the distribution of the $(\omega_\gamma)_\gamma$ near ω^+.

It is well-known that the band edge decay of the single site distribution modifies the Lifshitz tails (see e.g. [34]).

Before we start the description of the results, let us say that we will only describe results for lower band edges; in the case of internal gaps, similar results for upper band edges will also be true. Moreover, if E_- is the lower edge of a band, translating the periodic potential by a constant, we may assume that $E_- = 0$, which we do from now on.

Note also that, without restrictions, we may assume that $\omega^- = 0$ (one just needs to shift the periodic potential W) which we do from now on so that, as the random variables $(\omega_\gamma)_\gamma$ are not trivial, we get $\mathbb{E}(\omega_\gamma) = \mathbb{E}(\omega_0) > 0$.

We now discuss these different cases and the results that are known in each case.

3. The bottom of the spectrum

Let us first study the case of short range single site potentials.

3.1 *Short range potentials*

This is the case that has been studied the first and the most. Let us assume that $E_- = \inf(\Sigma) = 0$.

In [24,25], Lifshitz showed his heuristics not for the continuous Anderson model but for the Poisson model (the random potential is obtained by placing a nonnegative single site potential at every point of a Poisson process). He obtained the *Lifshitz tails*, namely

$$\log(N(E) - N(0)) = \log N(E) \underset{E \to 0^+}{\sim} -CE^{-d/2}. \tag{3.1}$$

Mathematical work on Lifshitz tails only appeared much later; the results were not obtained following the original Lifshitz heuristic but using Wiener integrals and the Donsker–Varadhan technique to estimate the Laplace transform of N (see [32,33]). This point of view is directly related to the study of Brownian motion among random obstacles (see [41]).

Proofs of Lifshitz tails at the bottom of the spectrum for the continuous Anderson model following the original heuristic of Lifshitz were developed in the 80s (see [12,13,29,40]). One does not obtain an asymptotic as precise as (3.1) but only the weaker version, namely

Theorem 3.1 [12,13,29,40]. *If one assumes that* (2.1) *holds*, *one has*

$$\log|\log N(E)| \underset{E \to 0^+}{\sim} -\frac{d}{2}\log E. \tag{3.2}$$

Proof. To understand the main ingredients needed for Lifshitz tails, let us just briefly sketch a proof of this result. This proof follows closely the original work done in [24,25,12,13,29,40]. To keep things as simple as possible, assume V has a compact support contained in C_0, the unit cell of $\mathbb{Z}^d$ and that $H_0 = -\Delta$.

Pick Λ_l the cube of side-length l centered at 0 in $\mathbb{R}^d$; let $H^D_{\omega,l}$ (resp. $H^N_{\omega,l}$) be the operator H_ω restricted to Λ_l with Dirichlet (resp. Neumann) boundary conditions. Also pick $E \in \mathbb{R}$ and define $\Theta^{D,N}_{\omega,l}(E)$ to be the eigenvalue counting functions for $H^{D,N}_{\omega,l}$ i.e. the number of eigenvalues of $H^{D,N}_{\omega,l}$ less than or equal to E. Then, the standard Dirichlet–Neumann bracketing (see e.g. [36,10,34]) tells us that

$$\frac{1}{\mathrm{Vol}(\Lambda_l)}\mathbb{E}(\Theta^D_{\omega,l}(E)) \le N(E) \le \frac{1}{\mathrm{Vol}(\Lambda_l)}\mathbb{E}(\Theta^N_{\omega,l}(E)). \tag{3.3}$$

The uniform relative boundedness of V_ω with respect to H ensures that, for $0 \le E \le 1$, one has $\Theta^N_{\omega,l}(E) \le C\ \mathrm{Vol}\,(\Lambda_l)$ (for some $C > 0$ independent of l and ω); so using Markov's inequality, from (3.3), we deduce

$$\begin{aligned}\frac{1}{\mathrm{Vol}(\Lambda_l)}\mathbb{P}(\{\text{ there exists an eigenvalue of } H^D_{\omega,l}(E)\ \le E\}) &\le N(E)\\ &\le C\mathbb{P}(\{\text{ there exists an eigenvalue of } H^N_{\omega,l}(E)\ \le E\}).\end{aligned}$$

This can be reformulated as

$$\begin{aligned}&\frac{1}{\mathrm{Vol}(\Lambda_l)}\mathbb{P}(\{\exists\varphi \in H^1_0(\Lambda_l),\ \|\varphi\| = 1,\ \langle H^D_{\omega,l}\varphi, \varphi\rangle \le E\}) \le N(E)\\ &\quad\le C\mathbb{P}(\{\{\exists\varphi \in H^1(\Lambda_l),\ \|\varphi\| = 1,\ \langle H^N_{\omega,l}\varphi, \varphi\rangle \le E\}).\end{aligned} \tag{3.4}$$

So to estimate $N(E)$ for E close to 0, we just have to choose l in such a way that (when taking the double logarithm) the left and right hand side of (3.4) asymptotically coincide.

Let us start with the lower bound which is the simpler bound. Therefore, pick $\alpha > 0$ small and $l = E^{-(1+\alpha)/2}$. The explicit knowledge of the eigenvalues for $H^D_{\omega,l}$ (see [36]) tells us that, for some function $\varphi_0 \in H^1_0$, $\|\varphi_0\| = 1$ (actually the eigenfunction associated to the lowest eigenvalue of the Dirichlet Laplacian on Λ_l), one has $\langle -\Delta\varphi_0, \varphi_0\rangle \leq CE^{1+\alpha}$ (where $C > 0$ is a constant). Hence, if we choose $\omega_\gamma \leq E^{-1-\alpha}$ for $\gamma \in \Lambda_l$, we get $\langle V_\omega\varphi_0, \varphi_0\rangle \leq CE^{1+\alpha}$ where $C > 0$ only depends on the sup-norm of the single site potentials. This proves that, for E sufficiently small and $l = E^{-(1+\alpha)/2}$, one has

$$\mathbb{P}(\{\{\exists\varphi \in H^1_0(\Lambda_l),\ \|\varphi\| = 1,\ \langle H_{\omega,l}\varphi, \varphi\rangle \leq E\}) \geq \mathbb{P}(\forall\gamma \in \Lambda_l,\ \omega_\gamma \leq E^{-1-\alpha}\}).$$

This last probability is easily estimated using the independence of the random variables. One gets

$$N(E) \geq E^{-(1+\alpha)d/2} \cdot \mathbb{P}(\omega_0 \leq E^{-1-\alpha})^{CE^{-(1+\alpha)d/2}}.$$

Using assumption (2.3) and (3.4), and letting α go to 0, one finally gets

$$\liminf_{E\to 0^+} \frac{\log|\log(N(E))|}{\log(E)} \geq -\frac{d}{2}. \tag{3.5}$$

The proof of the upper bound is more complicated. This is a general feature as the lower bound only requires finding the correct test function whereas for the upper bound, one has to show that all eigenfunctions (for energies in $[0, E]$) roughly behave in the same way.

This time, we pick $l = E^{-(1-\alpha)/2}$ where $0 < \alpha < 1$. Assume that $\varphi \in H^1(\Lambda_l)$, $\|\varphi\| = 1$ is such that $\langle H^N_{\omega,l}\varphi, \varphi\rangle \leq E$ i.e. such that

$$\langle -\Delta^N_l\varphi, \varphi\rangle + \langle V_\omega\varphi, \varphi\rangle \leq E. \tag{3.6}$$

Let φ_0 be the positive normalized eigenfunction associated to the first eigenvalues of $-\Delta^N_l$ i.e. to the eigenvalue 0 (φ is just the constant function equal to $(\mathrm{Vol}\,(\Lambda_l))^{-1/2}$ on Λ_l). Then, we can decompose $\varphi = a\varphi_0 + \psi$ where $\langle\varphi, \psi\rangle = 0$. One computes

$$\langle -\Delta^N_l\varphi, \varphi\rangle = a\langle -\Delta^N_l\varphi_0, \varphi\rangle + \langle -\Delta^N_l\psi, \varphi\rangle = \langle\psi, -\Delta^N_l\varphi\rangle = \langle -\Delta^N_l\psi, \psi\rangle.$$

Hence, by eq. (3.6) and as V_ω is non negative, we get

$$\langle -\Delta^N_l\psi, \psi\rangle \leq E. \tag{3.7}$$

On the other hand, an explicit computation (see e.g. [36]) shows that, $(-\Delta^N_l)_{\restriction_{\varphi_0^\perp}} \geq C^{-1}l^{-2} = C^{-1}E^{1-\alpha}$. Equation (3.7) then implies that

$$\|\psi\|^2 \leq CE^\alpha. \tag{3.8}$$

This in particular implies that $|a| = 1 + O(E^{\alpha/2})$. Now, fix $\beta \in (0, 1)$. Using the non-negativity of V_ω, we compute

$$\begin{aligned}\langle V_\omega\varphi, \varphi\rangle &= \|V_\omega^{1/2}\varphi\|^2 \\ &\geq \|aV_\omega^{1/2}\varphi_0\|^2 + \|V_\omega^{1/2}\psi\|^2 + 2\mathrm{Re}(a\langle V_\omega\varphi_0, \psi\rangle) \\ &\geq (1-\beta)\|aV_\omega^{1/2}\varphi_0\|^2 - (\beta^{-1} - 1)\|V_\omega^{1/2}\psi\|^2.\end{aligned}$$

Plugging this into (3.6) and using the non-negativity $-\Delta_l^N$, the boundedness of V_ω and estimate (3.8), we get

$$\langle V_\omega \varphi_0, \varphi_0\rangle \leq (a(1-\beta))^{-1}\left(E + (\beta^{-1}-1)CE^\alpha\right) \leq CE^\alpha. \tag{3.9}$$

On the other hand, as we assumed that the single site potential V has support in the unit cube of the lattice and as we know φ_0 explicitly, $\langle V_\omega\varphi_0, \varphi_0\rangle$ is equal to $(C\mathrm{Vol}(\Lambda_l))^{-1}$ $\Sigma_{\gamma\in\Lambda_l\cap\mathbb{Z}^d}\omega_\gamma$. In result, fixing $\beta = 2$, we have proved that, for any $\alpha \in (0,1)$, there exists $C > 0$ such that

$$\begin{aligned}&\mathbb{P}(\{\text{there exists an eigenvalue of } H^N_{\omega,l}(E) \ \leq E\}) \\ &\qquad\leq \mathbb{P}\left(\left\{\frac{1}{\mathrm{Vol}(\Lambda_l)}\sum_{\gamma\in\Lambda_l}\omega_\gamma \leq CE^\alpha\right\}\right).\end{aligned} \tag{3.10}$$

This last probability can be estimated using standard large deviation estimates (see e.g. [6,5]). For the sake of completeness, let us do this computation. Fix $\delta > 0$ small and let Λ be a large cube in $\mathbb{Z}^d$. Pick $t > 0$. Then, using Markov's inequality, one estimates

$$\mathbb{P}\left(\left\{\frac{1}{\mathrm{Vol}(\Lambda)}\sum_{\gamma\in\Lambda}\omega_\gamma \leq \delta\right\}\right) \leq \mathbb{E}\left(\mathrm{e}^{t(\delta-(1/\mathrm{Vol}(\Lambda))\sum_{\gamma\in\Lambda}\omega_\gamma)}\right). \tag{3.11}$$

Using the fact that the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are i.i.d., one gets

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{t(\delta-(1/\mathrm{Vol}(\Lambda))\sum_{\gamma\in\Lambda}\omega_\gamma)}\right) &= \mathrm{e}^{t\delta}\prod_{\gamma\in\Lambda}\mathbb{E}\left(\mathrm{e}^{-(t\omega_\gamma/\mathrm{Vol}(\Lambda))}\right) \\ &= \mathrm{e}^{t\delta}\,\mathrm{e}^{\mathrm{Vol}(\Lambda)\log\left(\mathbb{E}\left(\mathrm{e}^{-(t\omega_0/\mathrm{Vol}(\Lambda))}\right)\right)}.\end{aligned} \tag{3.12}$$

Taylor expanding e^t, one sees that $\mathbb{E}\left(\mathrm{e}^{-(t\omega_0/\mathrm{Vol}(\Lambda))}\right) = 1 - (t\mathbb{E}(\omega_0)/\mathrm{Vol}(\Lambda)) + O((t/\mathrm{Vol}(\Lambda))^2)$. Hence, there exists $\eta > 0$ such that for $0 < t \leq \eta\mathrm{Vol}(\Lambda)$, one has

$$\mathbb{E}\left(\mathrm{e}^{-(t\omega_0/\mathrm{Vol}(\Lambda))}\right) \leq \mathrm{e}^{-(t\mathbb{E}(\omega_0)/2\mathrm{Vol}(\Lambda))}.$$

Plugging this successively into eqs (3.12) and (3.11), we obtain that, for $0 < t \leq \eta\mathrm{Vol}(\Lambda)$, one has

$$\mathbb{P}\left(\left\{\frac{1}{\mathrm{Vol}(\Lambda)}\sum_{\gamma\in\Lambda}\omega_\gamma \leq \delta\right\}\right) \leq \mathrm{e}^{-t\frac{\mathbb{E}(\omega_0)-2\delta}{2}}. \tag{3.13}$$

So, as $\mathbb{E}(\omega_0) > 0$, if $4\delta < \mathbb{E}(\omega_0)$ and if $t = \eta_0\mathrm{Vol}(\Lambda)$, for some $\eta_0 > 0$ we get that

$$\mathbb{P}\left(\left\{\frac{1}{\mathrm{Vol}(\Lambda)}\sum_{\gamma\in\Lambda}\omega_\gamma \leq \delta\right\}\right) \leq \mathrm{e}^{-\eta_0\mathrm{Vol}(\Lambda)}.$$

Applying this to (3.10), using (3.4) and taking the double logarithm, we obtain for $\alpha \in (0,\ 1)$,

$$\limsup_{E\to 0^+}\frac{\log|\log(N(E))|}{\log(E)} \leq -\frac{d}{2}(1-\alpha).$$

This completes the proof of Theorem 3.1. □

Remark 3.1. The main ideas used in this proof goes back to the original work of Lifshitz [24,25]. First, for boxes that are not too large, the density of states at energy E is quite well approximated by the probability to find a state of energy less than E. As both the kinetic and the potential energy are non negative, they both must be less than E for that state. Smallness of the kinetic energy implies localization in momentum. Due to the uncertainty principle, this in turn implies that the wave has to be extended in some way in the position space. As the potential is ergodic, this implies that the potential energy of this wave is a space average of the potential. But, large deviation estimates tell that such averages have an exponentially small probability of being away from the average of the potential, hence, small in our case. This implies the exponential decay.

So we see that there is a crucial length scale governing the Lifshitz tail: the minimal size needed to find a state of kinetic energy less than E. This size depends on the form of the kinetic energy and, this explains why the bottom of the spectrum and internal edges behave differently (see § 4, especially Theorems 4.2, 4.4 and subsection 4.2.8). This size can also be interpreted as reflecting a local uncertainty principle.

3.2 *Long range potentials*

In this case, one proves that

Theorem 3.2 [12,13,29,40]. *If one assumes that* (2.2) *holds, one has*

$$\log|\log N(E)| \underset{E\to 0^+}{\sim} -\frac{d}{\nu-d}\log E. \tag{3.14}$$

Proof. For the sake of simplicity, we will assume that $V(x)=C(1+|x|)^{-\nu}$; the proof of the general case is not more difficult, just slightly more involved technically (see e.g. [13,10,34]).

As in the case of short range potentials, this proof is divided into two steps: a lower bound on the density of states and an upper bound. We will again use the inequality (3.4).

The proof of the lower bound is very similar to the short range potential case. As in this case, pick $l=E^{1/2+\alpha}$ ($\alpha\in(0,1)$). Then, if $\omega_\gamma\le E^{1+\alpha}$ for all $|\gamma|\le E^{1/(d-\nu)(1+\alpha)}$ and $E>0$ is sufficiently small, for $x\in\Lambda_l$, as the $(\omega_\gamma)_\gamma$ are bounded, we have

$$\begin{aligned}
0\le V_\omega(x)=C\sum_{\gamma\in\mathbb{Z}^d}\omega_\gamma(1+|x-\gamma|)^{-\nu}&\\
\le& CE^{1+\alpha}\sum_{|\gamma|\le E^{1/(d-\nu)-\alpha}}(1+|\gamma|)^{-\nu}+CK\sum_{|\gamma|>E^{1/(d-\nu)(1+\alpha)}}(1+|\gamma|)^{-\nu}\\
\le& CE^{1+\alpha}+CKE^{(d-\nu)/(d-\nu)(1+\alpha)}\le CE^{1+\alpha}.
\end{aligned}$$

This implies that in this case, one has

$$N(E)\ge E^{-(1+\alpha)d/2}\cdot\mathbb{P}(\omega_0\le E^{-1-\alpha})^{CE^{-(1+\alpha)d/(\nu-d)}}.$$

Taking (2.3) into account and letting $\alpha\to 0$, we obtain

$$\liminf_{E\to 0^+}\frac{\log|\log(N(E))|}{\log(E)}\ge-\frac{d}{\nu-d} \tag{3.15}$$

which completes the proof of the lower bound.

The proof of the upper bound is different from the short range potential case. As we will see, the kinetic energy does not play a role. The length scale l will be chosen later on during the proof.

As $-\Delta_l^N$ is non-negative, one has

$$\mathbb{P}(\{\{\exists\varphi \in H^1(\Lambda_l), \|\varphi\|=1, \langle H_{\omega,l}^N\varphi,\varphi\rangle \le E\}) \le \mathbb{P}(\{\exists x \in \Lambda_l,\ V_\omega(x) \le E\}). \tag{3.16}$$

So we estimate this last probability. Pick $\beta \in \Lambda_l \cap \mathbb{Z}^d$ and $x \in \Lambda_l$ such that $|x-\beta| \le 1/2$. Then, one has

$$V_\omega(x) = C\sum_{\gamma\in\mathbb{Z}^d}\omega_\gamma(1+|x-\gamma|)^{-\nu} \ge C\sum_{\gamma\in\mathbb{Z}^d}\omega_\gamma(1/2+|\beta-\gamma|)^{-\nu} =: A_\beta(\omega).$$

So we have proved that

$$\begin{aligned}\mathbb{P}(\{V_\omega > E\}) &\ge \mathbb{P}(\{\forall\gamma \in \Lambda_l \cap \mathbb{Z}^d;\ A_\gamma(\omega) > E\})\\ &\ge 1 - \sum_{\gamma\in\Lambda_l\cap\mathbb{Z}^d}\mathbb{P}(\{A_\gamma(\omega) \le E\}).\end{aligned}$$

Note that, as the $(\omega_\gamma)_\gamma$ are i.i.d, the random variables $A_\beta(\omega)$ are identically distributed. Hence

$$\mathbb{P}(\{V_{\omega,n} > E\}) \ge 1 - C(2l+1)^d\mathbb{P}(\{A_0(\omega) \le E\}).$$

Combining this with estimates (3.16) and (3.4), we obtain

$$N(E) \le C(2l+1)^d\mathbb{P}(\{A_0(\omega) \le E\}). \tag{3.17}$$

So we only need to estimate $\mathbb{P}(\{A_0(\omega) \le E\})$. To simplify the notations, set $u_\gamma = C(1/2+|\gamma|)^{-\nu}$. Using Markov's inequality, for any $\lambda > 0$, one obtains

$$\mathbb{P}(A_0(\omega) \le E) \le \mathbb{E}\left(\mathrm{e}^{\lambda(E-\sum\limits_{\gamma\in\mathbb{Z}^d}\omega_\gamma u_\gamma)}\right) \le \mathrm{e}^{\lambda E}\prod_{\gamma\in\mathbb{Z}^d}\mathbb{E}\left(\mathrm{e}^{-\lambda\omega_\gamma u_\gamma}\right). \tag{3.18}$$

Using the Taylor expansion of $x \mapsto \mathrm{e}^{-x}$, as the random variables $(\omega_\gamma)_\gamma$ are bounded, for λu_γ sufficiently small, one obtains

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{-\lambda\omega_\gamma u_\gamma}\right) \le 1 - \mathbb{E}(\omega_\gamma)\lambda u_\gamma + C(\lambda u_\gamma)^2 &\le 1 - \mathbb{E}(\omega_\gamma)\lambda u_\gamma(1-C\eta)\\ &\le \mathrm{e}^{-\overline{\omega_0}\lambda u_\gamma/2}.\end{aligned}$$

Hence, we have proved that there exists $\eta > 0$ such that, if $\lambda u_\gamma < \eta$, then

$$\mathbb{E}\left(\mathrm{e}^{-\lambda\omega_\gamma u_\gamma}\right) \le \mathrm{e}^{-\overline{\omega_0}\lambda u_\gamma/2} \text{ where } \overline{\omega_0} = \mathbb{E}(\omega_0) > 0. \tag{3.19}$$

Plugging (3.19) into (3.18), we compute

$$\log(\mathbb{P}(A_0(\omega) \le E)) \le \lambda E - \frac{\lambda\overline{\omega_0}}{2}\cdot\sum_{\substack{\gamma\in\mathbb{Z}^d\\ \lambda u_\gamma<\eta}} u_\gamma. \tag{3.20}$$

Assume that $l >> \lambda^{1/\nu}$; then, the definition of u_γ implies that, for some $C > 0$, one has

$$\sum_{\substack{\gamma\in\mathbb{Z}^d\\ \lambda u_\gamma<\eta}} u_\gamma \geq \sum_{\substack{\gamma\in\mathbb{Z}^d\\ C\lambda^{1/\nu}\leq|\gamma|}} (1+|\gamma|)^{-\nu} \geq \frac{1}{C}\lambda^{((d-\nu)/\nu)}.$$

Plugging this back into (3.20) and picking $\lambda = \rho E^{(\nu/(d-\nu))}$, we obtain

$$\log(\mathbb{P}(A_0(\omega) \leq E)) \leq -\rho(\overline{\omega_0}\rho^{d/\nu}/C - 1)E^{-(d/(\nu-d))} \leq -\frac{1}{C}E^{-(d/(\nu-d))}$$

for ρ sufficiently large.

If we pick e.g. $l \sim E^{-p}$ with $p > 0$, and plug estimate (3.2) into (3.17), we obtain, that for any $\alpha > 0$, one has

$$\limsup_{E\to 0^+} \frac{\log|\log(N(E))|}{\log(E)} \leq -\frac{d}{\nu-d}(1-\alpha).$$

Thus, we complete the proof of the upper bound and of Theorem 3.2.

Remark 3.2. The length scale that ruled the phenomenon in the short range case, did not play a role in the long range case. This essentially can be understood from the fact that, as the single site potentials are of longer range, the random potential is already obtained as an average; the effect of this averaging is more important than the effect of the averaging due to the spreading of the wave packet.

In this case, the Lifshitz exponent does not depend on the uncertainty principle i.e. of the kinetic energy. So one can expect that the same result should hold at general band edges for long range single site potentials. Theorem 4.1 asserts that this, indeed, is the case.

4. Internal band edges

In this case, much fewer results are known. In view of what is happening at the bottom of the spectrum, this is explained by the fact that for general periodic operators, very little is known about the band edge behavior. More precisely, if one considers a small random perturbation of a periodic operator, then it is sensible to assume that, near an open band edge of the periodic operator, the total random Hamiltonian will roughly be of the sum of a kinetic energy and a potential energy where the kinetic energy is given by the band edge dispersion relation of the periodic operators. Only very little is known about these functions in general; what makes the situation easier at the bottom of the spectrum is that the behavior of the band edge dispersion relation is well-known there (see [1,3]).

We will only describe results for the continuous Anderson model (the Poisson model does not have any internal gaps). As in the case of the bottom of the spectrum, the results are much simpler in the case of long range single site potentials.

4.1 *Long range potentials*

Let us start with the simplest case. We now assume that (2.2) holds. In this case, one proves that

Theorem 4.1 [20]. *Assume that* (H.1) *holds and that* V *satisfies* (2.2). *Assume the energy* $E = 0$ *is the bottom of a band of the almost sure spectrum* Σ. *Then, one has*

$$\lim_{E\to 0^+} \frac{\log|\log(N(E)-N(0))|}{\log E} = -\frac{d}{\nu - d}. \tag{4.1}$$

As announced, these results are the same as in the case of the bottom of the spectrum. We will now see that the situation is completely different for short range potentials.

4.2 *Short range potentials*

In the case of short range single site potentials, whether results are known depends heavily on the dimension d. In general dimension d, we only know of conditional results on Lifshitz tails. We start with those results and then turn to the special case of dimension $d = 2$ where we get more general results.

4.2.1 *In dimension* $d \geq 1$. In dimension 1, in [30], (3.2) is proved at all band edges. Again, this is explained by the fact that the behavior of the band function is well-known in this case (see e.g. [7,27,28] and references therein).

In dimension larger than 1, we start with a general result valid in any dimension. This result will also underline the importance of the band edge behavior of the dispersion relations. To state this result we need to introduce some additional objects.

4.2.2 *The periodic background operator.* This periodic background operator is a periodic Schrödinger operator that will play a crucial role in the study of Lifshitz tails. It depends on the spectral edge $E_\pm$ one is considering. It also depends on whether $E_\pm$ is the upper (resp. lower) edge (i.e. maximum (resp. minimum)) of a gap. Recall that the essential infimum (resp. supremum) of the random variables $(\omega_\gamma)_\gamma$ is ω^+ (resp. ω^-). Let H^+ (resp. H^-) be the periodic Schrödinger operator defined by $H^- = H + \omega^- \sum_{\gamma\in\mathbb{Z}^d} V_\gamma$ (resp. $H^+ = H + \omega^+ \sum_{\gamma\in\mathbb{Z}^d} V_\gamma$). Then one writes $H_\omega = H^- + V_\omega^-$ (resp. $H_\omega = H^+ + V_\omega^+$) where $V_\omega^- = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma^+ V_\gamma$ and $\omega_\gamma^+ = \omega_\gamma - \omega^-$ (resp. $V_\omega^+ = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma^- V_\gamma$ and $\omega_\gamma^- = \omega_\gamma - \omega^+$). The random variables $(\omega_\gamma^+)_\gamma$ (resp. $(\omega_\gamma^-)_\gamma$) are i.i.d. and non-negative (resp. non-positive). The results of [11] imply that

- E_- is the upper edge of a gap of Σ if and only if E_- is the upper edge of a gap of the spectrum of H^-.
- E_+ is the lower edge of a gap of Σ if and only if E_+ is the lower edge of a gap of the spectrum of H^+.

For the sake of definiteness, let us assume that E_- is the upper edge of a gap of Σ. Then, the correct background operator will be H^-.

4.2.3 *Floquet theory.* We recall some facts on the Floquet spectrum of H^- (see e.g. [36,23,38]). The Floquet spectrum of H^- is the spectrum of the differential operator H^- acting on $L^2_{\text{loc}}(\mathbb{R}^d)$ with quasi-periodic boundary conditions. For $\theta \in \mathbb{T}^* = \mathbb{R}^d/(\mathbb{Z}^d)^*$ (here $(\mathbb{Z}^d)^* = 2\pi\mathbb{Z}^d$ is the dual lattice of $\mathbb{Z}^d$ i.e. for $\gamma \in \mathbb{Z}^d$ and $\gamma^* \in (\mathbb{Z}^d)^*$, one has $\gamma\gamma^* \in 2\pi\mathbb{Z}$), consider the following eigenvalue problem on $L^2_{\text{loc}}(\mathbb{R}^d)$,

$$\begin{cases} H^-\varphi = E\varphi \\ \varphi(x+\gamma) = e^{i\gamma\theta}\varphi(x), \ \forall x \in \mathbb{R}^d, \ \forall \gamma \in \mathbb{Z}^d \end{cases}. \tag{4.2}$$

As H^- is elliptic, one knows that the eigenvalues of (4.2) are discrete; when repeated according to multiplicity, we denote them by $E_0(\theta) \leq E_1(\theta) \leq \cdots \leq E_n(\theta) \leq \ldots$. They are called the *Floquet eigenvalues* of H^-. These functions are Lipschitz continuous in the variable θ; when simple, they are even analytic in θ. Moreover, Weyl's law tells us that

$$E_n(\theta) \to +\infty \text{ as } n \to +\infty \text{ (uniformly in } \theta). \tag{4.3}$$

The spectrum of H^- is given by $\sigma(H^-) = \cup_{n\geq 0} E_n(\mathbb{T}^*)$. One can define $n(E)$, the integrated density of states of H_- in the same way as for H_ω (see (1.2)). One proves that (see [36,38])

$$n(E) = \sum_{j\geq 1} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^*} \mathbf{1}_{E_j(\theta)\leq E} \mathrm{d}\theta.$$

4.2.4 *A general result.* To simplify the notations, let us assume that $E_- = 0$. This can always be achieved by shifting the periodic background by a constant. One proves that

Theorem 4.2 [19]. *Assume* (H.1) *and that V satisfies* (2.1). *Then, one has*

$$\liminf_{E\to 0^+} \frac{\log|\log(N(E) - N(0))|}{\log E} \geq -\frac{d}{2}, \tag{4.4}$$

and

$$\lim_{E\to 0^+} \frac{\log|\log(N(E) - N(0))|}{\log E} = -\frac{d}{2} \Longleftrightarrow \lim_{E\to 0^+} \frac{\log(n(E) - n(0))|}{\log E} = -\frac{d}{2}. \tag{4.5}$$

Remark 4.1. The assertion (4.5) reflects what has been said above. Indeed, in the present case, the role of the kinetic energy is played by the periodic Schrödinger operator H^- and the density of states of this operator (when it satisfies the condition in (4.5)) controls the vanishing of the Floquet eigenvalues. This special situation is due to the fact that the quadratic vanishing is the quickest possible. In general the situation is more complicated and the sole asymptotic of the density of states of the underlying periodic operator is not enough to determine the decay of the density of states of the random operator (see Theorem 4.4 and the remark concluding subsection 4.2.8).

Of course, Theorem 4.2 also holds at the bottom of the spectrum; for short range single site potentials, it recovers all the cases previously described as it is known that the density of states of any periodic Schrödinger operator satisfies (4.5) at the bottom of the spectrum in any dimension and at all edges in dimension 1.

The proof of Theorem 4.2 relies on a new approximation scheme used to compute the density of states: the periodic approximation (see e.g. [19]). With Dirichlet–Neumann bracketing, one uses the fact that the Dirichlet (resp. Neumann) 'density of states' was a lower-bound (resp. upper-bound) of the true density of states. The crucial property of periodic approximations is that they approximate the true density of states exponentially well even if the finite chunk of space one is considering is not very large (see e.g [16]). This method has been used to prove results on the density of states of random operators in many other regimes, for Schrödinger operators with a random potential [22,21,37], for Schrödinger operators with random magnetic field [8], for random acoustic operators [31], etc.

4.2.5 *In dimension $d = 2$.* We now assume $d = 2$. In this case, we get more general results than in Theorem 4.2 as we will get results under no assumption on the density of states of the underlying periodic operator. We still assume $E_- = 0$.

4.2.6 *The density of states decays exponentially at band edges.* We will start with the less precise but more general result. Just for this paragraph, we do not assume anything about the decay of the single site potential (except for the summability condition stated in (H.1)). We prove

Theorem 4.3 [18]. *Assume* (H.1). *Let* 0 *be the lower edge of a band. Then, one has*

$$\limsup_{\substack{E\to 0\\ E\in\Sigma}} \frac{\log|\log|N(E)-N(0)||}{\log|E|} < 0. \tag{4.6}$$

Theorem 4.3 says that the integrated density of states decays at least exponentially fast at any band edge of Σ. But we do not get any information on the rate of decay. This is quite natural in view of Theorem 4.2 and of the fact that very little is known of the band-edge behavior of the Floquet eigenvalues. Of course, in the case of long range single site potentials, Theorem 4.1 already gives a better result.

4.2.7 *A precise estimate on the decay rate.* If we assume that the single site potentials are compactly supported which we do from now on, we are able to compute the rate of exponential decay of the integrated density of states.

Before we state our result, we need one more definition. Let H^- be the background operator introduced in § 4.2.2. Let E be in the spectrum of H^-. E will be *simple* if the set $\{p\in\mathbb{N};\ \exists\theta\in\mathbb{T}^*,\ E_p(\theta)=E\}$ is reduced to a single integer. Note that, by (4.3), this set of integers is always finite. An edge of a gap of Σ will be called simple if it is simple for H^- (resp. H^+) and it is the upper edge (resp. the lower edge) of a gap of Σ. By [17], we know that for a generic periodic potential, edges of gaps are simple.

We prove

Theorem 4.4 [18]. *Assume* (H.1) *and V the single site potential is compactly supported. Let* 0 *be the lower edge of a band of* Σ *and assume it is simple. Then, there exists* $0 < \alpha \le +\infty$ *such that*

$$\lim_{\substack{E\to 0\\ E\in\Sigma}} \frac{\log|\log|N(E)-N(0)||}{\log|E|} = -\alpha. \tag{4.7}$$

The exponent α is called the *Lifshitz exponent*. The computation of α is not obvious; the next section is devoted to its description.

4.2.8 *How to compute the Lifshitz exponent.* We recall that 0 is assumed to be the upper edge of a gap of Σ. Assume 0 is simple. Let $E_{n_0}(\cdot)$ be the unique Floquet eigenvalue taking the value 0. Define the set $\mathcal{S} = \{\theta\in\mathbb{T}^*;\ E_{n_0}(\theta)=0\}$. Then, there exists $\eta > 0$ such that

- For $n < n_0$, for all $\theta\in\mathbb{T}^*$, $E_n(\theta) < -\eta$.
- For $n > n_0$, for all $\theta\in\mathbb{T}^*$, $E_n(\theta) > \eta$.

This implies that, for θ in some neighborhood of $\mathcal{S}$, the Floquet eigenvalue $E_{n_0}(\theta)$ is simple, hence, the function $\theta\mapsto E_{n_0}(\theta)$ is real analytic in some neighborhood of $\mathcal{S}$. The Lifshitz exponent will depend on the way E_{n_0} vanishes at $\mathcal{S}$ and on the curvature of $\mathcal{S}$.

To describe it precisely, we need to introduce some objects from analytic geometry. If $\mathcal{E}$ is a set contained in the closed first quadrant in $\mathbb{R}^2$ then its *exterior convex hull* is the convex hull of the union of the rectangles $R_{xy} = [x, \infty) \times [y, \infty)$, where the union is taken over all $(x, y) \in \mathcal{E}$.

Pick $\theta_0 \in \mathcal{S}$ and consider the Newton diagram of E_{n_0} at θ_0, i.e.,

(1) Express E_{n_0} as a Taylor series at θ_0, $E_{n_0}(\theta^1, \theta^2) = \sum_{ij} a_{ij}(\theta^1 - \theta_0^1)^i(\theta^2 - \theta_0^2)^j$, $\theta = (\theta^1, \theta^2)$.
(2) Form the exterior convex hull of the points (i, j) with $a_{ij} \neq 0$. This is a convex polygon, called the *Newton polygon*.
(3) The boundary of the polygon is the *Newton diagram*.

The Newton decay exponent is then defined as follows: The Newton diagram consists of certain line segments. Extend each to a complete line and intersect it with the diagonal line $\theta^1 = \theta^2$. This gives a collection of points (a_k, a_k), one for each boundary segment. Take the reciprocal of the largest a_k and call this number $\tilde{\alpha}(E_{n_0}, \theta_0)$; it is the *Newton decay exponent*. Define $\alpha(E_{n_0}, \theta_0) = \min\{\tilde{\alpha}(E_{n_0} \circ T_0, \theta_0) : T_0(\cdot) = \theta_0 + T(\cdot - \theta_0),\ T \in SL(2, \mathbb{R})\}$.

Similarly, define $\alpha(E_{n_0}, \theta)$ if θ is any other point in $\mathcal{S}$, the zero set of E_{n_0}. Then, the *Lifshitz exponent* α is defined by

$$\alpha = \min_{\theta \in \mathcal{S}} \alpha(E_{n_0}, \theta). \tag{4.8}$$

The Lifshitz exponent α is positive as $\theta \mapsto \alpha(E_{n_0}, \theta)$ is a positive, lower semi-continuous function and $\mathcal{S}$ is compact.

To conclude this subsection, let us compare N to n, the density of states of the underlying periodic operator H_-. Up to a constant factor, $n(\varepsilon) - n(0)$ is the volume of $\{\theta \in \mathbb{T}^*;\ E_{n_0}(\theta) \leq \varepsilon\}$. From the equivalence (4.5) in Theorem 4.2, one may have conjectured that, at band edges N behaves roughly like $\mathrm{e}^{-1/n}$ (if one takes the double logarithm of both terms). We see here that this will not be true in general if the set $\mathcal{S}$ has some curvature. For example, if we assume that $E_{n_0}(\theta)$ vanishes to constant order q on a curve, and if the maximum order of vanishing of the curvature of the curve is m (thus non-vanishing curvature means $m = 0$) then $\alpha = (m + 3)/(q(m + 2))$ and the volume will have the exponent $1/q$.

4.2.9 *The main ideas of the proof of Theorem 4.4.* First, following the ideas developed in [19], we show that the study of the density of states of H_ω near 0 can be reduced to the study of the density of states of a discrete random operator. This discrete Hamiltonian is the sum of a kinetic energy part and a potential energy part. The kinetic energy part is made of the Floquet eigenvalues giving rise to the band starting at 0 for the background periodic operator H^-. The potential energy part is the random potential projected onto the space of states generating the band starting at 0. This discrete Hamiltonian, that we will call the reduced operator, can be realized as an operator acting on the torus $\mathbb{T}^*$. To perform this reduction, one needs to know that the edge 0 of Σ corresponds to an edge of a single periodic realization (namely H^-). Therefore we first need the positivity assumption of V and, second, we need to know that, when switching on the random potential (remember that H^- is now the unperturbed Hamiltonian), none of the spectrum below 0 passes above 0. In the present exposition, this is ensured by the fact that the gap below 0 is open in Σ and by the assumption that the random variables have connected support. Another natural setting where there cannot be any crossing is when one assumes that H^- has a gap and that the random perturbation V_ω^- is smaller than the length of the gap.

The next step is to show that, with good enough precision, we can estimate the integrated density of states of the reduced operator between 0 and ε by the probability that the reduced operator (restricted to some large cube) has an eigenvalue in the interval $[0, \varepsilon]$. If the band edge is simple, we then have to estimate the probability that, for $\delta > 0$ small, there exists $u \in L^2(\mathbb{T})$ such that the Fourier coefficients of u are supported in $[-M, M] \times [-M, M] \subset \mathbb{Z}^2$ and

$$\langle E_{n_0} \cdot u, u\rangle_{L^2(\mathbb{T}^*)} + \langle v_\omega u, u\rangle_{L^2(\mathbb{T}^*)} \leq \delta. \tag{4.9}$$

Here $E_{n_0} \cdot u$ denotes the multiplication by the Floquet eigenvalue E_{n_0} and v_ω denotes the reduced potential. Notice that both terms in (4.9) are non-negative. In the general case, the situation is somewhat complicated by the possible multiplicities.

At last, we estimate these probabilities. Carrying out this estimation is essentially a problem in random Fourier series; indeed, thanks to condition (2.3), for our purpose, we can replace the random variables $(\omega_\gamma)_\gamma$ by i.i.d. Bernoulli random variables.

Let f be a nonnegative real analytic function on $\mathbb{T}^*$, $\delta > 0$ and $M < \infty$ be parameters, S be a random subset of the square $[-M, M] \times [-M, M]$ (in $\mathbb{Z}^2$) with (say) density $1/2$ and $P(M, \delta)$ be the probability that there is a function $u \in L^2(\mathbb{T})$ such that

$$\int_{T^2} f(\theta)|u(\theta)|^2 < \delta \|u\|_2^2 \quad \text{and} \quad \text{supp}\,(\hat{u}) \subset S.$$

Thus the Fourier coefficients of u are to be supported on the random set, and u is to be mostly supported near the zero set of f. We study the behavior of the probability $P(M(\delta), \delta)$ as $\delta \to 0$, letting $M = M(\delta)$ go to ∞ in an appropriate manner. It turns out that $P(M(\delta), \delta)$ behaves roughly like $\mathrm{e}^{-\delta^{-\alpha}}$.

One can apply this bound taking f to be roughly the Floquet eigenvalue E_{n_0} to complete the proofs of Theorems 4.3 and 4.4, in a manner analogous to the one dimensional case treated in [16].

A Newton decay exponent similar to the one introduced above arises in problems concerning estimation of oscillatory integrals – see [42] and [35] – and, in [18], we show that it also controls the behavior of certain random Fourier series, in a way which to some extent is suggested by the results of [42] and [35].

Though in [18] we only study the case of random Schrödinger operators, the methods we develop can presumably be applied to many other families of random operators: random acoustic operators [31], random magnetic fields [8], etc.

References

[1] Agmon S, On positive solutions of elliptic operators with periodic coefficients in $L^2(\mathbf{R}^n)$, spectral results and extension to elliptic operators on Riemannian manifolds, in: Differential equations (eds) I Knowles and R Lewis (North-Holland: Birmingham) (1984)

[2] Barbaroux J-M, Combes J-M and Hislop P, Landau hamiltonians with unbounded random potentials. *Lett. Math. Phys.* **40(4)** (1997) 355–369

[3] Birman M Sh, Perturbations of operators with periodic coefficients, in: Schrödinger Operators: Standard and non-standard (Dubna) (1988)

[4] Carmona R and Lacroix J, Spectral Theory of Random Schrödinger Operators (Birkhäuser: Basel) (1990)

[5] Dembo A and Zeitouni O, Large deviation techniques and applications. (Boston: Jones and Bartlett Publishers) (1992)

[6] Deuschel J-M and Stroock D, Large deviations, Volume 137 of Pure and Applied Mathematics Academic Press (1989)

[7] Eastham M, The spectral theory of periodic differential operators (Edinburgh: Scottish Academic Press) (1973)

[8] Ghribi F, Asymptotique de Lifshitz pour des opérateurs de Schrödinger à champ magnétique aléatoire. Thèse Université Paris 13, (2000) en préparation

[9] Helffer B and Mohamed A, Asymptotic of the density of states for the Schrödinger operator with periodic electric potential. *Duke Math. J.* **92(1)** (1998) 1–60

[10] Kirsch W, Random Schrödinger operators, in Schrödinger Operators, A Jensen and H Holden (eds) number 345 in Lecture Notes in Physics, Berlin (1989) Springer Verlag, Proceedings, Sonderborg, Denmark (1988)

[11] Kirsch W and Martinelli F, On the spectrum of Schrödinger operators with a random potential, *Comm. Math. Phys.* **85** (1982) 329–350

[12] Kirsch W and Martinelli F, Large deviations and Lifshitz singularities of the integrated density of states of random hamiltonians, *Comm. Math. Phys.* **89** (1983) 27–40

[13] Kirsch W and Simon B, Lifshitz tails for the Anderson model, *J. Stat. Phys.* **38** (1985) 65–76

[14] Kirsch W, Stollmann P and Stolz G, Localization for random perturbations of periodic Schrödinger operators, *Random Oper. Stochastic Equ.* **6(3)** (1998) 241–268

[15] Klopp F, Localization for some continuous random Schrödinger operators, *Comm. Math. Phys.* **167** (1995) 553–570

[16] Klopp F, Band edge behaviour for the integrated density of states of random Jacobi matrices in dimension 1, *J. Stat. Phys.* **90(3-4)** (1998) 927–947

[17] Klopp F and Ralston J, Endpoints of the spectrum of periodic operators are generically simple, *Math. Appl. Anal.* **7(3)** (2000) 459–464

[18] Klopp F and Wolff T, Lifshitz tails for 2-dimensional random Schrödinger operators, in preparation (2000)

[19] Klopp F, Internal Lifshitz tails for random perturbations of periodic Schrödinger operators, *Duke Math. J.* **98(2)** (1999) 335–396

[20] Klopp F, Internal Lifshitz tails for Schrödinger operators with random potentials, *J. Math. Phys.* to appear in (2002)

[21] Klopp F, Precise high energy asymptotics for the integrated density of states of an unbounded random Jacobi matrix, *Rev. Math. Phys.* **12(4)** (2000) 575–620

[22] Klopp F and Pastur L, Lifshitz tails for random Schrödinger operators with negative singular Poisson potential, *Comm. Math. Phys.* **206(1)** (1999) 57–103

[23] Kuchment P, Floquet theory for partial differential equations, volume 60 of Operator Theory: Advances and Applications (Birkhäuser: Basel) (1993)

[24] Lifshitz I M, Structure of the energy spectrum of impurity bands in disordered solid solutions, *Sov. Phys. JETP* **17** (1963) 1159–1170

[25] Lifshitz I M, Energy spectrum structure and quantum states of disordered condensed systems, *Sov. Phys. Usp.* **7** (1965) 549–573

[26] Lifshitz I M, Gredeskul S A and Pastur L A, Introduction to the theory of disordered systems, (New-York: Wiley) (1988)

[27] McKean H and van Moerbeke P, The spectrum of Hill's equation, *Invent. Math.* **30** (1975) 217–274

[28] McKean H P and Trubowitz E, Hill's surfaces and their theta functions, *Bull. Am. Math. Soc.* **84(6)** (1978) 1042–1085

[29] Mezincescu G, Lifshitz singularities for periodic operators plus random potentials, *J. Stat. Phys.* **49** (1987) 1081–1090

[30] Mezincescu G, Internal Lifshits singularities for one dimensional Schrödinger operators, *Comm. Math. Phys.* **158** (1993) 315–325

[31] Najar H, Asymptotique de la densité d'états intégrée des modèles aléatoires continus, Thèse Université Paris 13 (2000) en préparation

[32] Nakao S, On the spectral distribution of the Schrödinger operator with random potential, *Jpn. J. Math.* **3** (1977) 117–139

[33] Pastur L, Behaviour of some Wiener integrals as $t \to +\infty$ and the density of states of the Schrödinger equation with a random potential, *Teor. Mat. Fiz.* **32** (1977) 88–95 (in Russian)

[34] Pastur L and Figotin A, Spectra of Random and Almost-Periodic Operators (Berlin: Springer Verlag) (1992)

[35] Phong D H and Stein E M, The Newton polyhedron and oscillatory integral operators, *Acta Math.* **179(1)** (1997) 105–152

[36] Reed M and Simon B, Methods of Modern Mathematical Physics, Vol IV: Analysis of Operators (New-York: Academic Press) (1978)

[37] Saad O, Comportement en grandes énergies de la densité d'états du modèle d'Anderson non borné, PhD thesis, Université de Paris 13, 1999, en préparation

[38] Sjöstrand J, Microlocal analysis for periodic magnetic Schrödinger equation and related questions, in: Microlocal analysis and applications, volume 1495 of Lecture Notes in Mathematics (Berlin: Springer Verlag) (1991)

[39] Skriganov M M, Proof of the Bethe–Sommerfeld conjecture in dimension 2, *Dokl. Akad. Nauk. SSSR* **248(1)** (1979) 39–42

[40] Stollmann P, Lifshitz asymptotics via linear coupling of disorder, *Math. Phys. Anal. Geom.* **2(3)** (1999) 279–289

[41] Sznitman A-S, Brownian motion, obstacles and random media (Berlin: Springer-Verlag) (1998)

[42] Varchenko V, Newton polyhedra and estimations of oscillatory integrals, *Funct. Anal. Appl.* **8** (1976) 175–196