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Diffusion controlled multiplicative process: typical versus average behaviour

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Abstract. We investigate the dynamics of the number of particles diffusing in a multiplicative medium. We show that the typical behaviour of the growth process is different from the average. We develop a new formalism to study the average growth process and extend it to the calculation of higher moments and finally of the probability distribution. We show that the fluctuations of the growth process increase exponentially with time. We describe the interesting features of the distribution.

Keywords. Random walk; multiplicative process; fluctuations; reaction kinetics; random growth process; diffusion.

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1. Introduction

The problem of the typical behaviour of a macroscopic quantity as compared to its average, has received considerable attention recently in the context of diffusion in disordered media (see Noskowicz and Goldhirsch (1988), Le Doussal (1989), Murthy and Kehr (1989), Kehr and Murthy (1990)). To motivate the basic issues involved we consider the following simple example.

Let Y denote the product of N identically distributed, positive definite, independent random variables, $x_1, x_2, \ldots, x_N$. Also, for simplicity, let x_i take only two values r_1 or r_2 with equal probability. It is clear that there are a total of 2^N configurations for the string $(x_1, x_2, \ldots, x_N)$. Amongst these, there are $n(Y)$ configurations, each leading to the same product Y. The value of Y for which $n(Y)$ is maximum is defined as the typical value, $\hat{Y}$. In other words, $\hat{Y}$ is the value of Y for which the entropy is maximum. In the present example, it is easily seen that $\hat{Y} = (r_1 r_2)^{N/2}$. On the other hand averaging over all configurations leads to $\langle Y \rangle = (r_1 + r_2)^N \cdot 2^{-N}$, which is different from $\hat{Y}$.

Let us consider the following experiment. Select randomly a string $(x_1, x_2, \ldots, x_N)$ by random sampling each x_i from its distribution. Calculate the value of Y for this string. Repeat the above independently M times and collect a set of values of Y, denoted by Y_i: $i = 1, M$. Let $\bar{Y}_M = (Y_1 + Y_2 + \cdots + Y_M)/M$. In the limit $M \to \infty$, $\bar{Y}_M = \langle Y \rangle$. However if M is not too large, $\bar{Y}_M$ is most likely to be close to $\hat{Y}$ rather than to $\langle Y \rangle$. Thus, the typical value $\hat{Y}$ represents more truly, the small sample average than does the true average. The task now is to obtain a 'working' expression for $\hat{Y}$.

Let $Z = \ln Y$. Then Z is a sum of N independent random variables $\{\ln x_i; i = 1, N\}$. For large N, by Central Limit Theorem (CLT), the distribution of Z is normal, with