Pramāṇa, Vol. 15, No. 6, December 1980, pp. 545–549, © Printed in India.

On the connectivity index for lattices of nonintegral dimensionality

DEEPAK DHAR
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

MS received 14 August 1980

Abstract. We define the connectivity index c for an infinite graph by the requirement that to disconnect a subset of at least V points from the rest of the graph requires the deletion of a minimum of $S(V)$ bonds where $S(V) \sim V^{(c-1)/c}$ for large V. For a d-dimensional hypercubical lattice with d integral, $c = d$. We construct explicit examples of lattices with nonintegral connectivity index c, $1 < c < \infty$. It is argued that the connectivity index is an important parameter determining the critical behaviour of Hamiltonians on these lattices.

Keywords. Graph theory; nonintegral dimension; connectivity index.

1. Introduction

In recent years, much attention has been devoted to studying the variation of critical exponents as a function of d, where d, the dimension of space, is treated as a continuously variable parameter. Wilson and Fisher developed the technique called ϵ-expansion which allows one to write critical exponents, say for Ising-like models as a power-series in ϵ, where $\epsilon = 4d$ (Fisher 1974; Wilson and Kogut 1974). These ϵ-expansion techniques have been pushed to quite high orders (Brezin *et al* 1974; Collet and Eckmann 1978). Similar series expansions in powers of ϵ, where the space dimension is 2-ϵ, 6-ϵ, 5-ϵ, etc. have been developed to describe a wide variety of phase transitions in different physical systems (Belavin and Yurishchev 1973; Harris *et al* 1976; Obukhov 1980). In quantum field theory the space dimension 4-ϵ has been introduced to regularise the ultraviolet divergences in the perturbation theory (Bollini and Giambiagi 1972).

Despite much work done dealing with the computational aspects of the ϵ-expansion technique (only a small part of which was cited above), its conceptual basis has remained quite obscure. Just what physical meaning may be assigned to these ϵ-expansions? We may argue that the appearance of ϵ as a continuous variable is a technical or mathematical artifice, and physically meaningful results correspond only to integral values of ϵ. This argument fails however, as the radius of convergence if these expansions (if they converge at all, presumably they are only asymptotic (Collet and Eckmann 1978) is certainly much less than 1.

In an earlier paper (Dhar 1977, hereafter referred to as I) we attempted to answer this question by explicitly constructing a class of lattices having a nonintegral value of the effective dimensionality. Defining a nearest-neighbour spring model on these lattices, we argued that if the fractional number of eigenmodes having frequency less than ω varies ω^d for small ω, then d should be identified as the (Fourier) dimension