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From Tanaka's formula to Ito's formula: The fundamental theorem of stochastic calculus

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Abstract. In this article we give a new proof of Ito's formula in $\mathbb{R}^n$ starting from the one-dimensional Tanaka formula. The proof is algebraic and does not use any limiting procedure. It uses the integration by parts formula, Fubini's theorem for stochastic integrals and essential properties of local times.

Keywords. Semi-martingales; Ito formula; Tanaka formula; local times.

1. Introduction

In this note we give a proof of Ito's formula starting from Tanaka's formula. Our approach was motivated by the proof of Ito's formula given in [1], and the results of [5], where we have proved and generalized the Tanaka formula, starting from first principles. In the case of dimension one, the basic method is well known, involving an integration in the space variable and then using the occupation density formula. We use a variation of this method, to go from Tanaka's formula to Ito's formula. In effect, we introduce an increasing process viz. the space integral of the local time and prove the occupation density formula and Ito's formula with this increasing process, which is then identified as the quadratic variation process by taking $f(x) = x^2$ in Ito's formula. The surprising fact is that the same method works in higher dimensions. Indeed, our formalism also yields a proof of the fundamental theorem of ordinary calculus, though only for twice differentiable functions: Let $x(t)$, $y(t)$ be continuous functions on $[0, \infty)$, of bounded variation on finite intervals and first let f be a C^2 function on $\mathbb{R}$ with compact support. Then

$$f(x_t) = f(x_0) + \int_0^t f'(x_s)\,\mathrm{d}x_s.$$

To prove this we first note that the 'ordinary' Tanaka formula holds for (x_t): If $a \in \mathbb{R}$, then

$$(x_t - a)^+ = (x_0 - a)^+ + \int_0^t I_{\{x_s > a\}}\mathrm{d}x_s.$$

This equation follows upon noting that the set $\{x_s > a\}$ is a union of disjoint open intervals of $[0, \infty)$. The integral in the RHS then reduces to a sum which together with the first term adds up to $(x_t - a)^+$. Integrating the ordinary Tanaka formula with respect to $f''(a)\,\mathrm{d}a$ and using Fubini's theorem we get the above expression for $f(x_t)$. We can then extend this equation for $f(x_t)$ to the arbitrary C^2 functions in the usual