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Spectral shift function and trace formula

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Abstract. The complete proofs of Krein's theorem on the spectral shift function and the trace formula are given for a pair of self-adjoint operators such that either (i) their difference is trace-class or (ii) the difference of their resolvents is trace-class. The proofs, essentially due to Krein, is based on Herglotz's theorem on the boundary value of the analytic functions whose imaginary part is non-negative on the upper half plane, and an almost optimal class of functions are obtained for which the trace formula is valid. Also an alternative method based on Weyl-von Neumann's theorem for self-adjoint operators, avoiding the complex function theory and inspired by Voiculescu's work, is given for the first case. Furthermore, some applications of the spectral shift function have been discussed.

Keywords. Spectral shift function; trace formula; Krein's theorem.

1. Introduction

Krein's spectral shift function and associated trace formulas [9, 18, 19, 20, 30] have been of considerable interest as an abstract mathematical statement as well as for various applications. The original proof of Krein (see for example [20]) uses analytic function theory and we use the same in § 2 and § 4. Voiculescu [28] gave a proof of the trace formula without using function theory for the case of bounded self-adjoint operators. We extend this method in § 3 to a pair of arbitrary self-adjoint operators with their difference trace-class. In the appendix we collect some of the necessary results from analytic function theory without proof as well as the definition and some properties of the perturbation determinant. Section 5 deals with some applications.

In this article, $\mathscr{H}$ will denote the Hilbert space we work in; $\mathscr{B}(\mathscr{H})$, $\mathscr{B}_1(\mathscr{H})$ and $\mathscr{B}_2(\mathscr{H})$ standing for the set of bounded, trace-class and Hilbert-Schmidt operators respectively. We shall often have H and H_0 as a pair of self-adjoint operators in $\mathscr{H}$ with $\sigma(H)$, $\sigma(H_0)$ their spectra; $\rho(H)$, $\rho(H_0)$ their resolvent sets with R_z and R_z^0 their resolvents and E_λ, E_λ^0 the associated spectral families. The symbols $\|\cdot\|$, $\|\cdot\|_1$ and $\|\cdot\|_2$ will denote operator norm, trace norm and Hilbert-Schmidt norm respectively, while $\operatorname{Tr} B$ will stand for the trace of a trace-class operator B.

In a finite dimensional Hilbert space the problem is easy to state and prove.

Theorem 1.1. *Let H and H_0 be two self-adjoint operators in a finite dimensional Hilbert space $\mathscr{H}$. Then there exists a unique real-valued bounded function ξ such that*

(i) $\xi(\lambda) = \operatorname{Tr}(E_\lambda^0 - E_\lambda)$, $\quad \lambda \in \mathbb{R}$,
(ii) $\int \xi(\lambda)\mathrm{d}\lambda = \operatorname{Tr}(H - H_0)$,