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A conjecture for some partial differential operators on $L^2(R^n)$

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Abstract. On $\mathscr{X} = L^2(R^n)$, let $Q = (Q_1, Q_2, \ldots, Q_n)$ and $P = (P_1, P_2, \ldots, P_n)$ be the operators given by $(Q_j f)(x) = x_j f(x)$, $P_j = -i\partial/\partial x_j$. For any C^∞ function $h: R^n \to R$ put $H_0 = h(P)$ and $H = H_0 + (1 + Q^2)^{-\delta}$, where $\delta > 1/2$. By the method of scattering theory we prove that H_{ac}, the absolutely continuous part of H is unitarily equivalent to H_0 when (*a*) $n = 1$ and (b) for $n \geqslant 2$, when h is in a large class of polynomials. It is conjectured that the results are true for any polynomial h. We use the techniques of Enss' method and the idea of bound states for momentum.

Keywords. Partial differential operators; scattering theory; Enss' method.

1. The conjecture and justification for the conjecture

Let $\mathscr{X} = L^2(R^n)$ be the Hilbert space of all square integrable functions on R^n. Let $Q = (Q_1, Q_2, \ldots, Q_n)$ and $P = (P_1, P_2, \ldots, P_n)$ be the position and momentum operators on $L^2(R^n)$ given by $(Q_j f)(x) = x_j f(x)$, and $(P_j f)(x) = -i(D_j f)(x)$, $D_j = \dfrac{\partial}{\partial x_j}$. Let $h: \mathbb{R}^n \to R$ be any C^∞ function such that h and all its derivatives have atmost polynomial growth. Put $H_0 = h(P)$. Let $W(x) = \langle x \rangle^{-1-\delta}$ for some $\delta > 0$ where $\langle x \rangle = (1 + x^2)^{1/2}$. Let $H = H_0 + W(Q)$. Let U_t and V_t be the free and total unitary evolution groups given by the self-adjoint operators H_0 and H viz

$$U_t = \exp[-itH_0], \qquad V_t = \exp[-itH]$$

Conjecture. Let h be any polynomial with $\sum_\alpha |D^\alpha h(\xi)|^2 \to \infty$ as $|\xi| \to \infty$. Let $g \in \mathscr{X}_{ac}(H)$, the absolutely continuous subspace for H. Then there exist $f_\pm$ in $\mathscr{X}$ such that $\| V_t g - U_t f_\pm \| \to 0$ as $t \to \pm\infty$. Consequently H on $\mathscr{X}_{ac}(H)$ is unitarily equivalent to H_0.

Remark 1. If $h(\xi) = \xi^2$ or h is elliptic or simply characteristic polynomial then the above result is known [E, Si, H2]. h is said to be simply characteristic if there exists constant K such that $\sum_\alpha |D^\alpha h(\xi)| \leqslant K[1 + |h(\xi)| + |\nabla h(\xi)|]$.

In this article we give an outline of the proof using ideas of [Mu1, 2] of the conjecture when (i) $n = 1$, the function h is very general such that $\langle Q \rangle^{-1} \langle h(P) \rangle^{-1}$ is compact and (ii) $n \geqslant 2$ and h is in a large class of polynomials. For $n = 1$, the proof given here is extremely simpler when compared with that given in [Mu1]. Throughout this article K with or without suffix will stand for a generic constant.