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Extended Kac–Akhiezer formulae and the Fredholm determinant of finite section Hilbert–Schmidt kernels

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Abstract. This paper deals with some results (known as Kac–Akhiezer formulae) on generalized Fredholm determinants for Hilbert–Schmidt operators on L_2-spaces, available in the literature for convolution kernels on intervals. The Kac–Akhiezer formulae have been obtained for kernels which are not necessarily of convolution nature and for domains in $\mathbb{R}^n$.

Keywords. Hilbert–Schmidt integral operator; Fredholm determinant; Kac–Akhiezer formula.

1. Introduction

The classical Fredholm determinant [2] of a given symmetric Hilbert–Schmidt integral (briefly H-S) operator T is an analytic function $D_T(\lambda)$ with the property that $\lambda(\neq 0)$ is a zero of $D_T(\lambda)$ if and only if $1/\lambda$ is an eigenvalue of T. We shall call any analytic function $f_T(\lambda)$, as a Fredholm determinant of T if its zeros are precisely reciprocals of the non-zero eigenvalues of T.

For convolution kernels such a Fredholm determinant was obtained by Kac [3], [4] and Akhiezer [1]. Their result is briefly summarized below:

Let $k(x)$ satisfy the following conditions:

1. $k(x)$ is a bounded, continuous, real valued function on $(-\infty, \infty)$,
2. $k(x) = k(-x)$,
3. $\hat{k} \in L_1(\mathbb{R}^1)$, where $\hat{k}$ is the Fourier transform of k,
4. $\int_{-\infty}^{\infty} |xk(x)| \mathrm{d}x < \infty$,

and

5. $\int_{-\infty}^{\infty} |xk(x)| \mathrm{d}x < 1/2$.

Then the Fredholm determinant $D_T(\lambda)$ of the symmetric H-S operator

$$T_\tau f = \int_0^\tau k(x-y) f(y) \mathrm{d}y$$

on $L_2[0, \tau]$ is given by

$$D_\tau(\lambda) = \prod_{j=1}^{\infty} [1 - \lambda\lambda_i(\tau)] = \exp\left[-\int_0^\tau \alpha(0, s, \lambda) \mathrm{d}s\right], \tag{1.1}$$