Proc. Indian Acad. Sci. (Math. Sci.), Vol. 104, No. 3, August 1994, pp. 543–548.

On polynomial isotopy of knot-types

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MS received 3 February 1993; revised 28 September 1993

Abstract. We have proved that every knot-type $\mathbb{R} \hookrightarrow \mathbb{R}^3$ can be uniquely represented by polynomials up to polynomial isotopy i.e. if two polynomial embeddings of $\mathbb{R}$ in $\mathbb{R}^3$ represent the same knot-type, then we can join them by polynomial embeddings.

Keywords. Non-compact knots; polynomial isotopy.

1. Introduction

Intuitively by a knot we mean a simple closed curve in $\mathbb{R}^3$ which is not the boundary of a smoothly embedded disc in $\mathbb{R}^3$, otherwise it is a *trivial knot*. Mathematically we define a knot as a smooth embedding of S^1 in $\mathbb{R}^3$ (or equivalently as a smooth embedding of S^1 in S^3). We identify a knot by its image and sometimes say that K is a knot, which means that K is the image of a smooth embedding of S^1 in $\mathbb{R}^3$. Two knots K_1 and K_2 given by embeddings ϕ_0 and ϕ_1 of S^1 in S^3 are called ambient isotopic if there exists an orientation preserving diffeomorphism $h:S^3 \to S^3$ such that $h(K_1) = K_2$, or equivalently there exists an isotopy $F:S^1 \times I \to S^3$ between ϕ_0 and ϕ_1.

Let K be a knot given by the embedding $\phi \equiv (\alpha, \beta, \gamma):S^1 \to \mathbb{R}^3$. In order to work with K we project it into a *suitable plane*. By suitable we mean that there are only finitely many *singular points* in the projected image and all of them are *ordinary double points* (i.e. the embedding followed by the projection is a *generic immersion*). Such a projection is called a *regular projection*. Suppose that K has a regular projection on the xy plane. Then for each double point in the projection there exist $t_1, t_2 \in S^1 (t_1 \neq t_2)$ such that $(\alpha(t_1), \beta(t_1)) = (\alpha(t_2), \beta(t_2))$. These double points are called *crossings* of the knot. At a given crossing for $t_1 < t_2$ if $\gamma(t_1) < \gamma(t_2)$ then it is an *under-crossing* and if $\gamma(t_1) > \gamma(t_2)$ it is an *over-crossing*. Knots which admit a regular projection are known as *tame knots* (for the precise definition of tame knot see [2]). Thus, for us a knot will always mean a tame-knot.

It is easy to see that each ambient isotopy class of knots in S^3 contains a knot given by a smooth embedding ϕ of S^1 in S^3 which maps the base point $(0,1) \in S^1$ to the base point $(0,0,0,1) \in S^3$ and whose derivative at $(0,1)$ is non-zero. Identifying $\mathbb{R}$ with $S^1 \backslash \{(0,1)\}$ and $\mathbb{R}^3$ with $S^3 \backslash \{(0,0,0,1)\}$, one observes that an embedding $\phi:S^1 \to S^3$ which is base point preserving and whose derivative at the given base point is non-zero, can be identified with a proper, smooth embedding $\tilde{\phi}:\mathbb{R} \to \mathbb{R}^3$ which is monotone (i.e. $\|\tilde{\phi}(x)\| \to \infty$ strictly monotonically as $|x| \to \infty$) outside a closed