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Rigidity problem for lattices in solvable Lie groups

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Abstract. The paper concerns rigidity problem for lattices in simply connected solvable Lie groups. A lattice $\Gamma \subset G$ is said to be rigid if for any isomorphism $\phi:\Gamma \to \Gamma'$ with another lattice $\Gamma' \subset G$ there exists an automorphism $\hat{\phi}:G \to G$ which extends ϕ. An effective rigidity criterion is proved which generalizes well-known rigidity theorems due to Malcev and Saito. New examples of rigid and nonrigid lattices are constructed. In particular, we construct: a) rigid lattice $\Gamma \subset G$ which is not Zariski dense in the adjoint representation of G, b) Zariski dense lattice $\Gamma \subset G$ which is not rigid, c) rigid virtually nilpotent lattice Γ in a solvable nonnilpotent Lie group G.

Keywords. Rigidity problem; Zariski density; lattice; Lie groups.

Introduction

Recall the basic definition. A lattice Γ in a Lie group G is said to be *rigid* (*weakly rigid*) in G if for any isomorphism $f:\Gamma \to \Gamma'$ of Γ with another lattice $\Gamma' \subset G$ (respectively for any automorphism $f:\Gamma \to \Gamma$) there exists an automorphism $\hat{f}:G \to G$ such that $\hat{f}|_\Gamma = f$.

The famous Mostow–Margulis–Prasad theorem (see, for instance, [8]) gives a sufficient condition for lattices in a broad class of semisimple Lie groups to be rigid. The solvable case seems to be less well studied. To formulate known results in this case let us give a classification of solvable groups mostly following [8].

The class of simply connected solvable Lie groups is divided into subclasses (N), (I), (R), (E), (A) in such a way that $(N) \subset (I)$, $(N) \subset (R) \subset (E) \subset (A)$ and $(I) \cap (A) = (N)$. Here (N) is the class of nilpotent groups, (R) and (I) are so-called classes of groups of "real" and "imaginary" types and (E) is the class of "exponential" groups. These classes are well known. We introduce the new class (A) named after L Auslander, whose paper [1] presents the first nontrivial example of Lie group of such a type with a lattice (see § 1 for exact definitions). This class is the maximal opposite class to the class (I):

Theorem 5.1. *Let Γ be a lattice in a connected simply connected solvable Lie group G. Then there exist normal connected subgroups G_I and G_A of types (I) and (A) respectively such that*

(1) $G = G_I G_A$,

(2) $\Gamma = (\Gamma \cap G_I)(\Gamma \cap G_A)$, *and*

(3) $G_I \cap G_A = N$, *where N is the nilradical of G.*