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Differential subordinations concerning starlike functions

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Abstract. Denote by $S^*(\rho)$, $(0 \leqslant \rho < 1)$, the family consisting of functions $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$ that are analytic and starlike of order ρ, in the unit disc $|z| < 1$. In the present article among other things, with very simple conditions on μ, ρ and $h(z)$ we prove the $f'(z)(f(z)/z)^{\mu-1} \prec h(z)$ implies $f \in S^*(\rho)$. Our results in this direction then admit new applications in the study of univalent functions. In many cases these results considerably extend the earlier works of Miller and Mocanu [6] and others.

Keywords. Differential subordination; univalent; starlike and convex functions.

1. Introduction

Let $\mathscr{A}$ denote the class of functions f that are analytic in the unit disc Δ and $f(0) = f'(0) - 1 = 0$, with $S^*(\rho)$, $0 \leqslant \rho < 1$, designating the subclass of $\mathscr{A}$ consisting of functions starlike (univalent) of order ρ. We denote by $\mathscr{P}$ the class of functions p that are analytic in Δ so that $p(0) = 1$.

From a result of Libera [5], it is known that if $p \in \mathscr{P}$ and if λ is a function defined on Δ with $\operatorname{Re} \lambda(z) > 0$ for $z \in \Delta$, then

$$p(z) + \lambda(z) z p'(z) \prec \frac{1+z}{1-z} \text{ implies } p(z) \prec \frac{1+z}{1-z}, \quad z \in \Delta, \tag{1}$$

where $\prec$ denotes subordination. This is an example of *non-autonomous differential subordination*; that is we allow functions of z to be present in addition to the terms $p(z)$ and $zp'(z)$. However the above result has been improved in [8] when $\operatorname{Re} \lambda(z) > \eta > 0$, $z \in \Delta$ and the sharp estimation has been obtained in [9] only when $\lambda(z) \equiv \alpha$, $\operatorname{Re} \alpha \geqslant 0$.

Using the Herglotz' representation theorem for functions with positive real part, one easily obtains that if p is analytic in Δ and $\alpha > 0$, then

$$\frac{1+z}{1-z} - \alpha \frac{z}{(1-z)^2} p(z) \prec \frac{1+z}{1-z} \text{ implies } p(z) \prec \frac{1+z}{1-z}, \quad z \in \Delta. \tag{2}$$

This provides another example of *non-autonomous differential subordination.*