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Deformations of complex structures on $\Gamma\backslash SL_2(\mathrm{C})$

C S RAJAN

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

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Abstract. Let G be a connected complex semisimple Lie group. Let Γ be a cocompact lattice in G. In this paper, we show that when G is $SL_2(\mathrm{C})$, nontrivial deformations of the canonical complex structure on X exist if and only if the first Betti number of the lattice Γ is non-zero. It may be remarked that for a wide class of arithmetic groups Γ, one can find a subgroup Γ' of finite index in Γ, such that $\Gamma'/[\Gamma',\Gamma']$ is finite (it is a conjecture of Thurston that this is true for all cocompact lattices in $SL(2,\mathrm{C})$).

We also show that G acts trivially on the coherent cohomology groups $H^i(\Gamma\backslash G,\mathcal{O})$ for any $i \geqslant 0$.

Keywords. Deformations; lattice; cohomology.

1. Introduction

Let M be a compact smooth manifold. We assume that M can be equipped with a hyperbolic structure. In ([3]), Johnson and Millson show that the space of deformations of 'marked conformal structures' on M has dimension at least r, where r is the largest number of disjoint, nonsingular, totally geodesic hypersurfaces in M. Such hypersurfaces are known to contribute to the first Betti number of M. Now, it is known that if the dimension of M is n, then M is diffeomorphic to $\Gamma\backslash SO(n,1)/K$, where Γ is a torsion-free cocompact lattice in $SO(n,1)$ and K is a maximal compact subgroup of $SO(n,1)$. When $n=3$, $SO(3,1)$ is locally isomorphic to $SL(2,\mathrm{C})$ and thus carries a complex structure. One can raise the question, whether there exists nontrivial deformations of the complex structure on $\Gamma\backslash SL(2,\mathrm{C})$, and if so whether the deformations are related to the 'topology of Γ'.

In a different direction, Matsushima raised the question whether the canonical complex structure on $\Gamma\backslash G$ is infinitesimally rigid, where G is a connected complex semisimple Lie group and Γ is an irreducible torsion-free cocompact lattice in G. In [8], Raghunathan showed that whenever G has no 3-dimensional components, the canonical complex structure on $\Gamma\backslash G$ is infinitesimally rigid. It is easy to extend this result to all G, provided G is not three-dimensional. We remark that when G is not three dimensional, the first Betti number of Γ is zero.

From these results, we are thus led to relating the 'topology of Γ' to the deformations of the complex structure on $\Gamma\backslash G$, where $G=SL(2,\mathrm{C})$. Our main result states that nontrivial deformations of the canonical complex structure on $\Gamma\backslash SL(2,\mathrm{C})$ exist if and only if the first Betti number of the cocompact torsion-free lattice Γ in $SL(2,\mathrm{C})$ is nonzero.