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Combinatorial manifolds with complementarity

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Abstract. A simplicial complex is said to satisfy complementarity if exactly one of each complementary pair of nonempty vertex-sets constitutes a face of the complex.

We show that if a d-dimensional combinatorial manifold M with n vertices satisfies complementarity then $d = 0$, 2, 4, 8 or 16 with $n = 3d/2 + 3$ and $|M|$ is a "manifold like a projective plane". Arnoux and Marin had earlier proved the converse statement.

Keywords. Combinatorial manifolds; complementarity.

1. Introduction

Recall that a *simplicial complex* K is a collection of nonempty sets (sets of *vertices*) such that all nonempty subsets of a member of the collection are again members. A member of K with $i+1$ vertices is called an *i-face* (or simplex of dimension i). For $\sigma \in K$ $\mathrm{Lk}(\sigma)$ (= *link* of σ) $:= \{\gamma \in K;\ \gamma \cap \sigma = \emptyset,\ \gamma \cup \sigma \in K\}$. A simplicial complex may be thought of as a prescription for the construction of a topological space by pasting together geometric simplexes. The topological space thus obtained from a simplicial complex K is called a *polyhedron* and is denoted by $|K|$. Let K_1 and K_2 be two simplicial complexes. A map $f: |K_1| \to |K_2|$ is called PL if there are subdivisions K'_1 and K'_2 of K_1 and K_2 respectively such that $f: K'_1 \to K'_2$ is simplicial. We write $|K_1| \approx |K_2|$ if $|K_1|$ and $|K_2|$ are PL homeomorphic. A simplicial complex K (respectively $|K|$) is called a *combinatorial d-manifold* (respectively PL *d-manifold*) if for every vertex v in K $Lk(v)$ is a $(d-1)$-dimensional combinatorial sphere.

In 1962, Eells and Kuiper [5] proved that a PL manifold M^d with PL Morse number $\mu(M^d) = 3$ has dimension $d = 0$, 2, 4, 8 or 16. If $d = 0$ M^d consists of three points. If $d = 2$ M^d is the real projective plane. For $d = 4$, 8 or 16, M^d is a simply connected cohomology projective plane over complex numbers, quaternions or Cayley numbers, respectively. Each of the manifolds of above type is called a *manifold like a projective plane*. This classification turned up in the 1987 paper [3] of Brehm and Kühnel on combinatorial manifolds with few vertices. Specifically, they proved that: Let M^d_n be a combinatorial d-manifold with n vertices,

(BK1) if $n < 3[d/2] + 3$ then $|M^d_n| \approx S^d$,
(BK2) if $n = 3(d/2) + 3$ and $|M^d_n| \not\approx S^d$ then $d = 2$, 4, 8 or 16 and $|M^d_n|$ must be a "manifold like a projective plane". Moreover for $d = 2$ $M^d_n = \mathbb{R}P^2_6$ and for $d = 4$ $M^d_n = \mathbb{C}P^2_9$.