

A note on a generalization of Macdonald's identities for A_ℓ and B_ℓ

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Abstract. Let $\eta(q)$ denote the Dedekind's η -function. Macdonald obtained identities for $\eta(q)^{\dim \mathfrak{g}}$ where \mathfrak{g} is complex simple finite dimensional Lie algebra. The aim of this paper is to obtain generalization of the above identities in the case of $\mathfrak{g} = A_\ell$ and B_ℓ . We also get new formulas for the generating functions of the Ramanujan's τ -function and ψ_α -functions.

Keywords. Macdonald's multivariable identities; Dedekind's eta function;

1. Introduction

Let $\eta(q)$ denote Dedekind's η -function. Macdonald [7] obtained a formula for $\eta(q)^{\dim \mathfrak{g}}$ for every complex simple Lie algebra \mathfrak{g} which gives a generalization of the Jacobi's expansion for $\eta(q)^3$. These formulas are some specializations of the Macdonald's multivariable identities [7]. Many other identities involving Dedekind's η -function were also obtained by Lepowsky [4].

We shall need the following preliminaries. Let \mathfrak{g} denote the simple finite dimensional Lie algebra of the type A_ℓ or B_ℓ ; \mathfrak{h} denotes its Cartan subalgebra; Let Δ , Δ_+ and Δ_+^\vee be the root system, the set of positive roots and the positive dual roots respectively of \mathfrak{g} and ρ (respectively ρ^\vee) be the half-sum of the roots in Δ_+ (resp. Δ_+^\vee). W and M are the Weyl group of \mathfrak{g} and the lattice spanned over \mathbb{Z} by the long roots of \mathfrak{g} respectively. Let h and g denote the Coxeter and dual Coxeter numbers of \mathfrak{g} respectively.

Let \langle, \rangle denote the pairing of the elements in \mathfrak{h} and \mathfrak{h}^* (cf. [3]). We introduce the following notations:

$$\Delta_m^\vee = \{\alpha \in \Delta_+^\vee / \langle \rho, \alpha \rangle \equiv 0 \pmod{m}\},$$

$$d_m(\lambda) = \prod_{\alpha \in \Delta_m^\vee} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}, \quad (\lambda \in \mathfrak{h}^*).$$

The aim of this paper is to prove the following:

Theorem 1. For $\ell \geq 1$, let $m \leq \ell + 1$ be any divisor of $\ell + 1$. Then

$$\eta(q^m)^{(\ell+1)^2/m} \eta(q)^{-1} = \sum_{\alpha \in M} d_m((\ell+1)\alpha)^{(1/2(\ell+1))|\rho+(\ell+1)\alpha|^2}$$