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Bertini theorems for ideals linked to a given ideal

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Abstract. We prove a generalization of Flenner's local Bertini theorem for complete intersections. More generally, we study properties of the 'general' ideal linked to a given ideal. Our results imply the following. Let R be a regular local Nagata ring containing an infinite perfect field k, and $I \subset R$ is an equidimensional radical ideal of height r, such that R/I is Cohen-Macaulay and a local complete intersection in codimension 1. Then for the 'general' linked ideal J_α, R/J_α is normal and Cohen-Macaulay.

The proofs involve a combination of the method of basic elements, applied to suitable blow ups.

Keywords. Linkage; local Bertini theorem; Nash blow up; r-fold basic elements.

1. Introduction

Let R be a Noetherian local ring, and $I \subset R$ an ideal of height r. This paper is concerned with questions of the following type: if J is a "sufficiently general" ideal in R which is geometrically linked to I (see § 6 for definition), then what "good properties" does the ring R/J inherit from R, or from R/I?

Part of the problem is to make precise the notion of "sufficiently general". We will work with the following natural notion, motivated by the standard one in algebraic geometry. We assume that our local ring R contains an infinite field k. Suppose $g_1, \ldots, g_N$ is a set of generators for I. Then we can consider the set of r-tuples of k-linear combinations $\{\Sigma_j \alpha_{ij} g_j | 1 \leqslant i \leqslant r\}$ as the (k-valued) points of the affine space $\mathbf{A}^{rN}$. For any such r-tuple $\alpha \in \mathbf{A}^{rN}$, let $\mathfrak{F}_\alpha = \Sigma_i R(\Sigma_j \alpha_{ij} g_j)$ be the ideal generated by the r-tuple. Define $J_\alpha = (\mathfrak{F}_\alpha : I)$. We will say that some property holds *for the general* J_α (or just *for general* α) if the property is valid for all points α in a non-empty Zariski open subset of $\mathbf{A}^{rN}$.

For $r = 1$, this notion of "general" is used by Flenner in formulating some of his local Bertini theorems. For example our Theorem 2 (see section 4) is a generalisation of the following theorem of Flenner ([F], Sätz 4.6), apart from the fact that Flenner's proof works for any set of generators of I.

Theorem (Flenner). *Let (A, m) be a Noetherian local k-algebra, where k is a field of characteristic 0. Let $I \subset A$ be an ideal, with a given set of generators $x_1, \ldots, x_N$. Then for general $\alpha \in k^N$, the quotient ring $R/x_\alpha R$ is regular on $D(I) \cap reg(R)$. Here $x_\alpha = \Sigma_j \alpha_j x_j$.*

We show, by an example in § 4, that if k has characteristic $p > 0$, then Theorem 2 above does not hold for an arbitrary set of generators of I.