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Kolmogorov's existence theorem for Markov processes in C^* algebras

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Dedicated to the memory of Professor K G Ramanathan

Abstract. Given a family of transition probability functions between measure spaces and an initial distribution Kolmogorov's existence theorem associates a unique Markov process on the product space. Here a canonical non-commutative analogue of this result is established for families of completely positive maps between C^* algebras satisfying the Chapman-Kolmogorov equations. This could be the starting point for a theory of quantum Markov processes.

Keywords. Completely positive map; Markov process; GNS principle.

1. Introduction

Let $(X_i, \mathscr{F}_i)$, $i = 0, 1, 2, \ldots$ be Polish measurable spaces and let $P_i(x_i, dx_{i+1})$ be a transition probability from $(X_i, \mathscr{F}_i)$ to $(X_{i+1}, \mathscr{F}_{i+1})$ for each i. Given a probability measure μ on $(X_0, \mathscr{F}_0)$ it follows from Kolmogorov's extension theorem that there exists a unique probability measure P_μ on the infinite product space $(\Omega, \mathscr{F}) = \bigotimes_{i=0}^{\infty}(X_i, \mathscr{F}_i)$ such that, for every finite n, its projection or marginal distribution P_μ^n in $\bigotimes_{i=0}^{n}(X_i, \mathscr{F}_i)$ is given by

$$P_\mu^n(E_0 \times E_1 \times \cdots \times E_n) = \int_{E_0 \times E_1 \times \cdots \times E_n} \mu(dx_0) P_0(x_0, dx_1) P_1(x_1, dx_2) \cdots P_n(x_{n-1}, dx_n) \tag{1.1}$$

for all $E_i \in \mathscr{F}_i$, $i = 0, 1, 2, \ldots, n$. The probability space $(\Omega, \mathscr{F}, P_\mu)$ describes the Markov process with initial distribution μ and transition probability $P_i(\cdot,\cdot)$ for transition from a state at time i to a new state at time $i + 1$. This can be described in a * algebraic language as follows. Denote by $\mathscr{A}_i$ the commutative * algebra of all complex valued bounded measurable functions on $(X_i, \mathscr{F}_i)$. Introduce the positive unital operator $T(i, i+1)$: $\mathscr{A}_{i+1} \to \mathscr{A}_i$ by

$$(T(i, i+1)g)(x_i) = \int g(x_{i+1}) P_i(x_i, dx_{i+1}).$$

For any $i \leqslant k$ define $T(i, k)$: $\mathscr{A}_k \to \mathscr{A}_i$ by

$$T(i,k) = \begin{cases} \text{identity} & \text{if } i = k, \\ T(i, i+1)\, T(i+1, i+2) \cdots T(k-1, k) & \text{if } i < k. \end{cases}$$